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A REVIEW OF GEOMETRICALLY EXACT BEAM FORMULATIONS AND THEIR ISOGEOMETRIC IMPLEMENTATIONS: FROM FINITE ROTATIONS TO ISOGEOMETRIC ANALYSIS

Abstract:

This paper presents a review of the development of geometrically exact beam theories for the analysis of large displacements and finite rotations of curved beams. Since finite rotations evolve on the nonlinear manifold $SO(3)$ rather than in a vector space, their consistent parametrization, interpolation, and update are essential to maintain objectivity, avoid path dependence, and achieve robust nonlinear solution behavior. The paper surveys finite element developments for geometrically exact beams, focusing on Simo-Reissner formulations and comparison with Kirchhoff-Love models. The review concludes with isogeometric formulations for both beam theories, addressing locking phenomena.

Keywords: curved beam, finite rotation, geometrically exact theory, IGA, locking.

ПРЕГЛЕД ГЕОМЕТРИЈСКИ ТАЧНИХ ГРЕДНИХ ТЕОРИЈА И ЊИХОВА ИЗОГЕОМЕТРИЈСКА ИМПЛЕМЕНТАЦИЈА: ОД КОНАЧНИХ РОТАЦИЈА ДО ИЗОГЕОМЕТРИЈСКЕ АНАЛИЗЕ

Сажетак:

Рад представља преглед развоја геометријски тачних гредних теорија за анализу великих помјерања и коначних ротација кривих греда. Будући да коначне ротације леже на нелинеарној многострукости $SO(3)$, а не у вектрском простору, њихова конзистентна параметризација, интерполација и ажурирање су неопходни како би се очувала објективности, избјегла зависност од путање и постигло робусно понашање нелинеарног рјешења. Рад испитује развој методе коначних елемената за геометријски тачне греде, с нагласком на Симо-Рајснерове формулације и поређење са Кирхоф-Лав моделима. Преглед се завршава изогеометријским формулацијама за обје гредне теорије, обрађујући и феномене локинга.

Кључне ријечи: крива греда, коначна ротација, геометријски егзактна теорија, ИГА, локинг.

1. INTRODUCTION

Beam is the fundamental mechanical model for slender structural bodies in which the longitudinal dimension greatly exceeds the transverse ones. Curved beam elements have become essential in aerospace, civil, and offshore engineering, as well as in emerging fields such as biomechanics, electronics, and optics [1], [2]. Their geometric elegance, favorable load-carrying characteristics, and efficient material utilization further contribute to their widespread adoption in contemporary structural design. In these applications, flexible beams frequently undergo large displacements and finite rotations while maintaining small strains, and the presence of initial curvature introduces strong coupling between axial, bending, and torsional responses. Classical beam theories, based on small-displacement and small-rotation assumptions, are therefore inadequate, motivating the development of geometrically exact (GE) frameworks that retain the full nonlinear kinematics without linearization.

A fundamental challenge in this setting is the mathematical structure of finite rotations, which form a nonlinear manifold rather than a vector space [3], [4]. The choice of rotation parametrization has a direct influence on the efficiency and robustness of the resulting formulation, and no single parametrization is free of singularities or other limitations. This in turn affects the interpolation of rotational degrees of freedom and the construction of objective and path-independent finite element discretizations. In particular, standard interpolation procedures for translational degrees of freedom cannot be directly extended to rotational degrees of freedom, since the discretization must remain invariant under superposed rigid-body motions (i.e., it must not generate spurious strains under a pure rigid translation or rotation). To allow accurate and efficient beam formulation, both variation and linearization must be done consistently. The present paper provides a review of key developments in this field, from the mathematical foundations of finite rotations to recent finite element and isogeometric formulations.

The review begins with the geometry of the $SO(3)$ manifold and the most widely used rotation parametrization strategies. Classical shear-rigid and shear-deformable beam theories are briefly recalled before tracing the development of GE beam theories and their finite element implementations. The discussion covers the seminal contributions of Reissner, Antman, and Simo, as well as the early numerical frameworks that laid the foundation for modern geometrically exact beam analysis. The main emphasis is placed on static formulations, as this is where most contributions are introduced and compared, although extensions to dynamics, stability, beam contact, and advanced material behavior are also indicated. Particular attention is given to GE Simo-Reissner formulations, together with a review of the development of GE Kirchhoff-Love models. A brief overview of alternative finite element approaches for the analysis of geometrically nonlinear beams capable of large displacements and rotations is also provided. The review concludes with isogeometric beam formulations, addressing locking phenomena and the strategies developed to overcome them, providing an overview of this active research area.

2. DESCRIPTION OF FINITE ROTATIONS

The mathematical study of finite rotations has a long history in classical mechanics. A fundamental result, known as *Euler's theorem* [4], [5], states: “*any rigid motion leaving a point fixed may be represented by a rotation about a suitable axis passing through that point*”. This theorem underlies all rotation representations used in computational mechanics and confirms that three parameters are both necessary and sufficient to describe an arbitrary orientation in three-dimensional space.

Mathematically, a finite rotation is represented by a proper orthogonal tensor $\mathbf{R}$, which is a linear transformation of three-dimensional Euclidean space that preserves lengths and angles. The set of all such tensors forms the *special orthogonal group*, defined as

$$SO(3) := \{\mathbf{R}: E^3 \rightarrow E^3 \text{ linear } | \mathbf{R}^T\mathbf{R} = \mathbf{I}, \det\mathbf{R} = 1\} \tag{1}$$

where E^3 denotes the three-dimensional Euclidean vector space. The first condition ensures preservation of lengths and angles, and the second distinguishes proper rotations from reflections. A rotation acts on a vector $\mathbf{v} \in \mathbb{R}^3$ by mapping it to a new vector $\mathbf{v}' = \mathbf{R}\mathbf{v}$, preserving its length while changing its direction. $SO(3)$ is a smooth curved manifold and not a flat linear space [3], [6]. Rotation tensors can be represented minimally by three parameters, though no single three-parameter global representation exists that is free of singularities [7].

Rotations differ fundamentally from displacements. Displacements belong to a linear vector space and can be freely added and scaled. Rotations, by contrast, are nonlinear objects that cannot be added like ordinary vectors. Moreover, their composition is order-dependent, meaning that two successive rotations applied in different sequences generally produce different final orientations. This property is known as non-commutativity [4], [8] and is expressed as

$$\mathbf{R}_1\mathbf{R}_2 \neq \mathbf{R}_2\mathbf{R}_1. \tag{2}$$

At any point on $SO(3)$, the curved manifold can be locally approximated by a flat linear space, called the tangent space at that point. This tangent space captures all admissible infinitesimal rotations at a given rotational configuration. The tangent space at the identity **I** is known as the Lie algebra $so(3)$, and consists of all skew-symmetric tensors, as defined by

$$so(3) := \{\widehat{\mathbf{A}}: E^3 \rightarrow E^3 \text{ linear } |\widehat{\mathbf{A}}^T = -\widehat{\mathbf{A}}\}. \tag{3}$$

Since $SO(3)$ is a Lie group, the tangent spaces at all points of the manifold share the same algebraic structure and can all be identified with $so(3)$ [4]. This means that incremental rotations can only be added if they belong to the same tangent space [7]. Tangent vectors at different points of the manifold can be transported between tangent spaces, through push-forward and pull-back operations [4], [9]. This allows linear operations on rotational quantities to always be referred back to the fixed space $so(3)$, regardless of the current rotational configuration.

The connection between this linear tangent space and the curved manifold $SO(3)$ is established by the *exponential map*. For a rotation vector $\boldsymbol{\varphi} = \varphi\,\mathbf{n}$, where φ is the rotation angle and **n** is the unit rotation axis, the exponential map takes the closed form known as the *Rodrigues rotation formula* [8]:

$$\mathbf{R} = \exp(\widehat{\boldsymbol{\varphi}}) = \mathbf{I} + \frac{\sin\varphi}{\varphi}\,\widehat{\boldsymbol{\varphi}} + \frac{1-\cos\varphi}{\varphi^2}\,\widehat{\boldsymbol{\varphi}}^2 \tag{4}$$

where the hat accent, $\widehat{\boldsymbol{\varphi}}$, denotes the skew-symmetric tensor of its axial vector $\boldsymbol{\varphi} \in \mathbb{R}^3$. This formula maps a three-component vector to the corresponding rotation tensor. The correspondence between rotation vectors and rotation tensors is unique for rotation angles $\varphi < \pi$, but the same rotation can also be represented by infinitely many rotation vectors differing by multiples of 2π [4], [10]. A discussion of the origin of this formula can be found in [11].

The geometric picture is illustrated in Figure 1. The rotation manifold $SO(3)$ is represented as a curved surface, where each point represents a specific orientation of a rigid body. The two points shown, the identity **I** and a general rotational state **R**, each possess their own tangent space providing a local linear approximation to the manifold. The exponential map $\exp(\widehat{\boldsymbol{\varphi}})$ connects a skew-symmetric tensor $\widehat{\boldsymbol{\varphi}}$ in the tangent space at the identity, $T_I SO(3)$, to the corresponding point on the manifold. A rotation from **I** to **R** follows a geodesic, the shortest path lying entirely on the manifold, whereas a straight-line interpolation between two rotation tensors leaves $SO(3)$. This is the fundamental source of difficulty when rotational fields are interpolated in finite element discretizations, as discussed in Section 5.

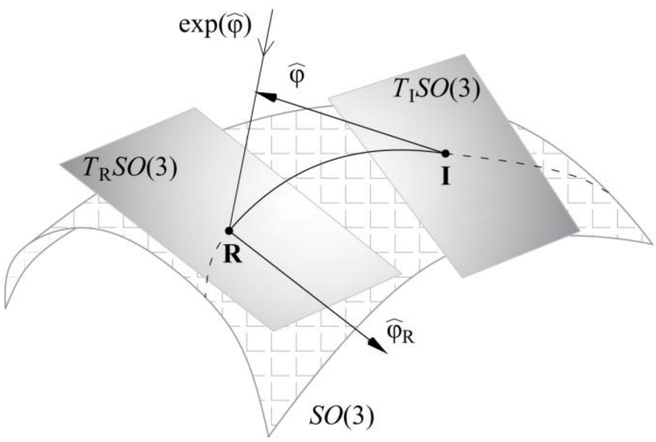

Figure 1. The rotation manifold $SO(3)$ with tangent spaces $T_I SO(3)$ and $T_R SO(3)$, skew-symmetric tensors $\widehat{\boldsymbol{\varphi}}$ and $\widehat{\boldsymbol{\varphi}}_R$, and the exponential map $exp(\widehat{\boldsymbol{\varphi}})$ (created by author).

In iterative nonlinear solution procedures, rotation updates must respect the curved structure of $SO(3)$. Given a current rotation $\mathbf{R}$ and an incremental rotation vector $\Delta\boldsymbol{\varphi}$, the updated rotation is obtained through the *multiplicative update* rule [7], [12]:

$$\mathbf{R}_{\text{new}} = exp(\widehat{\boldsymbol{\varphi}}_{new}) = \exp(\widehat{\Delta\boldsymbol{\varphi}})\,\mathbf{R} = \exp(\widehat{\Delta\boldsymbol{\varphi}})\exp(\widehat{\boldsymbol{\varphi}}) \neq \exp(\widehat{\boldsymbol{\varphi}} + \widehat{\Delta\boldsymbol{\varphi}}) \tag{1}$$

$$\rightarrow\ \widehat{\boldsymbol{\varphi}}_{new} \neq \widehat{\boldsymbol{\varphi}} + \widehat{\Delta\boldsymbol{\varphi}}$$

This multiplicative structure replaces the additive update used in linear theories and is one of the key characteristics of GE formulations. The tangent operator $\mathbf{T}(\boldsymbol{\varphi})$ arises because the rotation vector $\boldsymbol{\varphi}$ and the rotation tensor $\mathbf{R}$ belong to different spaces, and a linear mapping between their variations is needed for consistent linearization. It plays the role of a Jacobian of the exponential map and reduces to the identity for zero rotation, recovering the standard additive update algorithm of infinitesimal-rotation theories.

3. PARAMETRIZATION OF FINITE ROTATIONS

As established in Section 2, rotations form the Lie group $SO(3)$, a smooth curved manifold that is non-Euclidean and non-commutative. A central consequence of this structure is that no smooth minimal three-parameter representation can cover the entire rotation group without exhibiting either analytical singularities or global non-uniqueness [4], [10]. This constraint shapes the choice of parametrization in finite element formulations. Historical accounts of the development of rotation representations in applied mathematics and mechanics are given in [11], [13], [14].

Rotation parametrizations can be classified along two complementary lines. The first concerns the number of parameters employed. The most direct representation uses the full rotation matrix, with nine components subject to six orthogonality constraints. Although globally valid, enforcing these constraints complicates implementation and increases computational cost. Euler angles reduce the description to three parameters with a clear geometric interpretation but depend on the chosen rotation sequence and exhibit singular configurations known as *gimbal lock*, limiting their robustness in nonlinear analyses. A comprehensive overview of classical parametrizations is given in [15]. The second classification distinguishes vectorial from non-vectorial parametrizations. Vectorial parametrizations describe rotations through a geometric vector in $\mathbb{R}^3$, often called rotational quasi-coordinates, and align naturally with the Lie algebra structure of $SO(3)$. Non-vectorial parametrizations, such as Euler angles and quaternions, cannot be expressed as a single geometric vector. This distinction, discussed in [16], [17], provides a framework for comparing parametrizations with respect to singularity behavior, computational efficiency, and suitability for a finite element implementation.

Among minimal non-vectorial parametrizations, *Euler parameters*, commonly known as *quaternions*, have gained particular importance in computational mechanics. Although they use four parameters subject to one normalization constraint, they are globally free of singularities and numerically stable. Introduced by Hamilton in 1843, quaternions offer algebraically simpler rotational operations compared to matrix-based representations, which explains their widespread use in computational applications involving large rotations. Their advantages in nonlinear finite element analysis were demonstrated in [12], [18], [19], and one of the early consistent finite element implementations was provided by Cardona and Géradin [20]. The non-uniqueness of quaternions is topological: $\boldsymbol{q}$ and $-\boldsymbol{q}$ always represent the same physical rotation, so each rotation corresponds to exactly two quaternion representations [10], [21]. Further discussion can be found in [3], [8], [22].

Within the class of vectorial parametrizations, a unified framework can be established through a scalar generating function of the rotation angle, as developed by Bauchau and Trainelli [16]. Several classical three-parameter representations arise as particular choices within this framework, allowing a consistent comparison of admissible range, singular behavior, and linearization properties.

Among minimal vectorial parametrizations, *Rodrigues parameters*, also known as *Cayley-Gibbs-Rodrigues parameters*, were used prominently by Argyris and co-workers [8], [23], [24], [25], where rotations were interpreted through semitangential measures. This parametrization has rational transformation formulas, making it computationally convenient, but exhibits a singularity at $\varphi = \pi$, where the underlying Cayley transformation becomes undefined.

The *rotation vector* formulation, based on the exponential map, became the dominant choice in GE beam [12], [18], [19], [26]. It uses the Lie algebra of $SO(3)$ and the exponential map to relate skew-symmetric tensors to rotation matrices, providing a minimal description well-suited for the linearization of the spatially discretized weak form. Its main limitation appears at $\varphi = \pi$, where the rotation axis $\boldsymbol{n}$ and $-\boldsymbol{n}$ produce indistinguishable rotations and the rotation vector loses uniqueness. This is a global property of parametrization rather than a breakdown of the local mapping [6], [7].

An alternative minimal vectorial representation is the *conformal* rotation vector, also known as the *Wiener-Milenković* or *modified Rodrigues* parametrization. It is obtained by projecting the unit quaternion onto $\mathbb{R}^3$ through a stereographic mapping, yielding rational update formulas and moving the singularity to $\varphi = 2\pi$. This more favorable distribution of singularities often improves numerical conditioning in large-rotation computations [15], [27], [28]. In computational mechanics and multibody dynamics, this parametrization has been used alongside exponential coordinates and quaternions where robust tangent operators are required [16], [17].

Other vectorial parametrizations, such as sine-based or linearized variants, have been proposed for specific purposes but are suited only for restricted rotation ranges and sacrifice global regularity for algebraic simplicity [16].

A fundamentally different strategy that avoids explicit parametrization altogether by representing the orientation of the cross-section directly through director vectors rather than through rotation parameters is developed by Betsch and Steinmann [29], [30]. They reformulated the weak form of the governing equations so that rotations are represented through director fields rather than classical rotation parameters, avoiding direct interpolation of rotational unknowns and preserving objectivity of the discrete formulation.

The parametrizations described above apply to rotations at a single point. In finite element discretizations, however, rotations must also be interpolated between nodes, which introduces additional challenges arising from the nonlinear structure of $SO(3)$, as discussed in the following section.

4. DEVELOPMENT OF GEOMETRICALLY EXACT BEAM THEORIES AND EARLY FINITE ELEMENT FORMULATIONS

The classical theory of beams originates from the works of Euler and Bernoulli in the eighteenth century, where slender structures were modeled under the assumption that cross-sections remain plane and perpendicular to the neutral axis during deformation. This hypothesis neglects transverse shear deformation and is accurate only for very slender beams in bending-dominated responses.

To address this limitation, Timoshenko and Ehrenfest introduced a refined beam theory in the early twentieth century. By allowing independent rotations of the cross-section and assuming a constant transverse shear strain through the cross-section, the Timoshenko-Ehrenfest theory provides improved accuracy for moderately thick beams and vibration analyses. Since the assumed shear stress distribution is not fully consistent with three-dimensional elasticity, shear correction factors were later introduced to compensate for this discrepancy [31].

Although shear-deformable beam theories represent an important improvement over classical shear-rigid models, both theories are limited to small rotations and moderate displacements. This restricts their applicability to structures undergoing arbitrarily large rotations and displacements, particularly for initially curved beams, where curvature and torsion of the reference axis introduce coupling between axial, bending, and torsional responses. These limitations motivated the development of geometrically nonlinear beam formulations capable of describing large motions in a consistent and frame-invariant manner.

The first rigorous attempts to describe beams undergoing large displacements and rotations can be traced back to Kirchhoff's theory of rods [32], which extended Euler's elastica to arbitrarily curved slender bodies while assuming rigid cross-sections and neglecting transverse shear deformation. Subsequent refinements by Love incorporated small axial stretching [33], leading to what is commonly referred to as the Kirchhoff-Love rod theory. Despite their conceptual importance, these formulations neglected shear deformation and were restricted to the small-rotation regime.

A decisive step beyond classical assumptions was made by Reissner. His planar formulation [34] incorporated transverse shear deformation into a nonlinear beam theory capable of large displacements and rotations. The extension to spatially curved beams was first addressed in [35] by generalizing Kirchhoff-type equations to include force-deformational effects alongside bending and twisting moments. This formulation was later re-derived in a simpler form in [36], where a new description of rotational displacement states was introduced and applied to helical deformations. Crisfield and Jelenić [37] noted that Reissner[36] employed a simplified spatial parametrization of the rotation matrix, which facilitated the derivation of rotational strain measures but compromises strict geometric exactness. Nevertheless, Reissner's contributions provided an essential foundation for later fully GE formulations.

A related and influential contribution was provided by Antman [38], [39], who formulated a general theory of nonlinearly elastic rods extending the Kirchhoff-Love framework to include both axial extension and transverse shear deformation. His formulation admitted general nonlinear constitutive relations between stress resultants and strain measures without requiring a strain energy density function, providing a broad continuum framework for rod mechanics.

Building on the works of Reissner and Antman [34], [38], [39], Simo formulated a fully geometrically exact three-dimensional beam theory derived from nonlinear continuum mechanics [12]. Rotations were treated as elements of the Lie group $SO(3)$, yielding strain-configuration relations that remain exact for arbitrarily large displacements and rotations. The corresponding finite element formulation was developed by Simo and Vu-Quoc [18] for static and dynamic analyses, followed by an implicit time-integration scheme formulated directly on $SO(3)$ [40], giving rise to what is commonly referred to as the Simo-Reissner beam theory. While geometrically exact in its kinematics, the formulation does not account for large strains when linear elastic constitutive laws are assumed. Later works incorporated cross-sectional warping [41] and provided a rigorous justification for the symmetry of the tangent stiffness matrix as the true Hessian of the potential energy [42].

Cardona and Géradin [43] introduced tangential transformation, a linear mapping that projects rotation increments from different tangent spaces onto a common tangent space where standard vector operations are valid. In the same work, they distinguished three rotation update approaches: the Eulerian, Total Lagrangian, and Updated Lagrangian. Their Updated Lagrangian formulation alleviates singularity issues associated with total rotation parametrizations and ensures that incremental rotation vectors remain within a consistent tangent space. Related contributions were made by Iura and Atluri [44], [45], who developed a variationally consistent formulation for finitely stretched and rotated spatial beams and highlighted the importance of consistent rotation update strategies.

While these early works established the essential framework for GE beam analysis, the discrete treatment of rotational degrees of freedom required further refinement, as discussed in the following section.

5. FINITE ELEMENT FORMULATIONS FOR GEOMETRICALLY EXACT BEAMS

The finite element formulations reviewed in this section build on the continuum framework of Section 4 and focuses on the consistent discretization of rotational degrees of freedom on the $SO(3)$ manifold.

5.1. GEOMETRICALLY EXACT SIMO-REISSNER BEAMS

Within the GE Simo-Reissner framework, subsequent finite element developments focused on improving how finite rotations are represented and updated at the discrete level. These formulations mainly differ in the choice of rotation parametrization, the interpolation of rotational variables, and the update procedure adopted in nonlinear analyses [46], [47], [48], [49], [50], [51].

Early contributions focused on practical finite element implementations of geometrically nonlinear beam models. Crisfield [19] developed a formulation for three-dimensional beams based on incremental rotation vectors with a consistently derived tangent stiffness matrix. Ibrahimbegović and co-authors developed finite element formulations for Reissner-type beams, including three-dimensional curved beam elements for large rotations [47]. The numerical treatment of finite rotations using vector-like parametrizations was investigated in [48] and complemented by a systematic discussion of rotation parameter choices and their implications for nonlinear finite element formulations [26]. For broader background on nonlinear finite element theory and implementation, see [52].

A fundamental observation was made by Crisfield and Jelenić [37], [53], who showed that directly interpolating rotation parameters, or their incremental and iterative changes, can destroy the objectivity of discrete strain measures even when the underlying continuum strains are objective. They further showed that interpolating total rotations leads to non-objective but path-independent formulations, whereas interpolating incremental or iterative rotations results in formulations that are both non-objective and path-dependent. To overcome this, they proposed an interpolation based on relative rotations: the relative rotation between nodal triads is computed, its rotation vector is extracted and interpolated in a common tangent space, and the rotation field is reconstructed via the exponential map on $SO(3)$. This eliminates spurious strains under rigid-body motion and provides a clear basis for objective finite element implementations of GE beams.

A broad range of interpolation and update strategies for finite rotations has been proposed, differing in how the nonlinear structure of $SO(3)$ is handled at the discrete level. Parametrization-based approaches remain widely used due to their compact implementation, most commonly relying on rotation vectors [16] or unit quaternions[54], [55], [56]. These are employed either as total rotation parameters [26], [48], [52] or through incremental [19], [43], [51] and iterative update procedures [47], [50]. Since interpolation is performed in linear spaces, such schemes do not generally respect the geometry of $SO(3)$ and may compromise objectivity or induce path dependence [22], [37], [53]. Relative-rotation schemes construct the interpolated rotation field from rotations between adjacent nodal frames, ensuring consistency with rigid-body motion [53], [55], [57], [58]. Director-based discretizations represent cross-section orientation through interpolated director fields, preserving frame invariance without explicitly interpolating rotation parameters [29]. Lie-group formulations treat rotations directly as elements of $SO(3)$ or $SE(3)$ and define interpolation and update procedures in a manifold-consistent manner [59], [60]. Strain-based approaches avoid explicit rotation interpolation by discretizing curvature or rotational strain measures directly, enforcing objectivity at the strain level [61], [62]. The geometric aspects of rotation updates, including singularities and tangent-space considerations, were studied in [7], [63], while stability of spatial beams and frames was addressed through direct computation of critical equilibrium states [64]. In general, formulations that explicitly account for the manifold structure of $SO(3)$ show improved robustness, particularly for large rotations and higher-order discretizations [59], [65].

Mixed and enhanced formulations have also been explored to improve convergence and robustness under complex loading and large rotations. Mixed finite element formulations for GE beams have shown improved iterative convergence and reduced sensitivity to locking [66], [67].

All approaches reviewed above share the common goal of ensuring a consistent and objective discrete treatment of finite rotations. The key finding is that numerical inconsistencies such as spurious strains originate not from the continuum theory but from the interpolation of rotational variables at the discrete level [37], [53], [68]. A kinematically distinct class of GE beam theories based on the Kirchhoff constraint is reviewed in the following section.

5.2. GEOMETRICALLY EXACT KIRCHHOFF-LOVE BEAMS

Geometrically exact finite element formulations for Kirchhoff-Love beams have received less attention than Simo–Reissner formulations. The main difficulty is that the Kirchhoff constraint couples the rotational field directly to the displacement field, requiring C^1 continuity of the interpolation, which standard Lagrangian finite elements cannot easily satisfy.

Early contributions to GE Kirchhoff-Love beam elements were made by Weiss [69], [70] and Boyer [70], though both were limited to initially straight beams. Armero and Valverde [71], [72] examined objectivity and geometric exactness of existing Kirchhoff-type elements more systematically, showing that early formulations combined C^0-continuous Lagrange polynomials for axial displacements with C^1-continuous Hermite polynomials for transverse displacements, which led to a loss of objectivity. Membrane locking was also addressed in these works through assumed strain fields and reduced integration.

The formulation by Meier et al. [73] represents the first Kirchhoff beam element capable of simultaneously satisfying geometric exactness, objectivity, and three-dimensional large-deformation analysis for beams with arbitrary initial geometries. They further argued that Kirchhoff elements offer advantages over Simo-Reissner elements at high slenderness ratios, where shear modes can cause numerical ill-conditioning and shear locking. A subsequent study [74] addressed membrane locking in slender curved beams by proposing a new interpolation strategy for the axial strain field and comparing it with assumed natural strains and reduced integration. Beam-to-beam contact was later treated using a torsion-free beam formulation [75], while a detailed comparison of Kirchhoff-Love and Simo-Reissner theories and their finite element implementations is provided in [2].

These formulations later served as a basis for isogeometric discretizations of GE beams, as reviewed in Section 5.4.

5.3. ALTERNATIVE FORMULATIONS: CO-ROTATIONAL AND ABSOLUTE NODAL COORDINATE APPROACHES

Several alternative formulations have been proposed for the finite element analysis of geometrically nonlinear beams, among which the co-rotational formulation [19], [52], [76], [77], [78], [79] and the Absolute Nodal Coordinate Formulation (ANCF) [65], [80], [81], [82], [83] are the most widely used.

The co-rotational approach decomposes the total deformation of a beam element into a rigid-body part, associated with large rotations and translations, and a local deformation part measured relative to a frame that follows the element during motion. This decomposition allows classical linear theories to be applied within the local frame, while geometric nonlinearity is fully captured through the transformation between the co-rotational and global coordinate systems [76], [84], [85]. A notable advantage of this approach is that existing linear finite elements can be reused within the co-rotational framework, and the resulting formulations naturally preserve strain invariance under rigid-body motion and yield path-independent solutions for conservative problems [78], [79], [86]. The approach has also been applied to instability analysis of three-dimensional frame structures, where co-rotational beam elements with a rotation vector parametrization of finite rotations have been used to trace post-critical equilibrium paths [87].

The co-rotational concept has also influenced GE formulations, and the boundary between the two approaches is not always sharp [2], [76]. Jelenić and Crisfield [53] drew on ideas from co-rotational theory to propose an interpolation strategy that restores objectivity and path-independence while preserving the exactness of the strain measure. Building on this, Magisano et al. [67] proposed a formulation in which incremental nodal rotation vectors define co-rotational nodal rotations, which are then interpolated for strain evaluation within a GE framework. The key result is that all properties of the GE continuum model, exact strain measure, objectivity, and path-independence, are preserved at the discrete level for any mesh and interpolation order, rather than recovered only asymptotically upon mesh refinement as in standard co-rotational approaches. Additive updates and a symmetric tangent stiffness matrix under conservative loading are also retained.

The ANCF, originally introduced by Shabana [80], [88], takes a fundamentally different approach by describing beam motion through global nodal position and slope coordinates rather than rotational degrees of freedom. Developed originally for the dynamic analysis of flexible multibody systems, the formulation employs standard shape functions to interpolate the three-dimensional displacement field, yielding a constant mass matrix and eliminating centrifugal and Coriolis inertia forces. The ANCF element library has expanded considerably to include beam, plate, shell, and solid elements, with applications in robotics, vehicle dynamics, and aerospace structures [89], [90]. Ongoing research has focused on alleviating locking phenomena inherent to ANCF elements, including transverse shear and Poisson locking, through reduced integration, enhanced formulations, and the introduction of additional gradient vectors [91], [92].

5.4. ISOGEOMETRIC FORMULATIONS AND LOCKING

Isogeometric analysis (IGA) was originally motivated by the need to bridge the gap between geometric modelling and numerical analysis by defining the basis functions on initial geometry, and then using the same functions for the approximation of unknown fields [93], [94]. Compared to classical finite elements, IGA allows higher polynomial orders while maintaining a comparable number of degrees of freedom, and the resulting smooth solutions require fewer integration points [95], [96]. The higher interelement continuity of spline basis functions brings an additional benefit in the context of beam formulations, as it naturally alleviates certain locking phenomena that commonly affect standard finite element discretizations. This has made IGA an attractive framework for the development of higher-order beam elements with improved accuracy and reduced sensitivity to locking.

An important development within the IGA framework is the isogeometric collocation method, which solves the strong form of the governing equations at discrete collocation points rather than through numerical integration of a weak form. This approach significantly reduces computational cost compared to Galerkin-based IGA while retaining the smoothness properties of spline basis functions. Early applications of IGA collocation to beam problems demonstrated its potential for naturally avoiding shear locking in Timoshenko-Ehrenfest formulations [97], [98], [99], representing a computationally efficient alternative to conventional Galerkin discretizations. The extension of isogeometric collocation to GE shear-deformable beams was first addressed by Marino [100] through a displacement-based formulation. A subsequent contribution [101] introduced both displacement-based and mixed formulations for beams with arbitrary initial curvature, with the mixed approach shown to be completely locking-free regardless of the approximation degree.

Weeger et al. [102] extended isogeometric collocation to the Simo-Reissner beam model and proposed a mixed formulation in which internal forces and moments are discretized independently from the kinematic unknowns, effectively alleviating shear locking in slender beams. A subsequent contribution [103] extended this framework to beams with spatially varying geometric and material parameters. Dynamic analysis of GE beams using isogeometric collocation was addressed in [104], [105], where explicit and implicit time integration schemes were developed.

Further contributions extended Simo-Reissner isogeometric beam formulations in several directions. Choi et al. [106] combined a GE shear-deformable beam formulation with an adjoint-based procedure for a configuration design sensitivity analysis. Within a total Lagrangian setting, Vo and co-workers developed isogeometric Timoshenko-type formulations [107], [108] and later proposed locking-free variants for geometrically nonlinear planar analysis [109]. Building on earlier mixed finite element developments [66], Magisano et al. [110] extended the approach to the isogeometric setting through a patch-wise formulation that eliminates locking while substantially reducing the number of degrees of freedom. Adopting a solid-like kinematic description, Choi et al. [111] introduced extensible director vectors to capture in-plane cross-sectional deformations and enable the direct use of three-dimensional constitutive laws, with Poisson locking alleviated through an enhanced assumed strain method. This framework was extended to frictionless beam-to-beam contact with unconstrained directors in [112], and computational efficiency of mixed variants was subsequently improved through selectively reduced-degree basis functions for stress resultant fields [113]. Recent dynamic contributions include a fully explicit isogeometric collocation formulation [114] and objective mixed extensible-director formulations for nonlinear beam elastodynamics [115]. Several of these formulations also incorporate material nonlinearity, ranging from hyperelastic three-dimensional constitutive laws [111], [112] to elasto-visco-plasticity with softening in mixed collocation [116] and materially nonlinear elastodynamics [115].

Early isogeometric formulations for Kirchhoff-Love space rods were developed by Greco and Cuomo, who introduced models for spatial beam assemblies and addressed mixed formulations and consistent tangent operators [117], [118], [119], [120]. Building on this foundation, Bauer et al. [121] developed the first nonlinear isogeometric formulation for spatially curved Kirchhoff-Love beams and demonstrated its applicability to large deformation problems.

More recent contributions have focused on multi-patch configurations and structural assemblies, motivated by the reduced number of degrees of freedom compared to shear-deformable elements. Vo et al. [122] developed a geometrically nonlinear multi-patch isogeometric Euler-Bernoulli formulation using the smallest-rotation mapping to describe finite cross-section rotations. Greco et al. subsequently extended their formulation to beam assemblies [123] and updated Lagrangian analysis [124]. The influence of initial curvature on the convergence properties of the nonlinear solution procedure was investigated in [125].

Radenković and Borković contributed a series of isogeometric Bernoulli-Euler beam formulations of increasing complexity. In the terminology adopted by these authors, the Bernoulli-Euler designation refers to the kinematic theory known as the Kirchhoff-Love beam theory. Rotation-free formulations for linear plane beams were developed for static [126] and dynamic [127] analysis, while Radenković et al. [128] extended the approach to spatially curved beams in the linear static case, introducing torsional rotation as an independent degree of freedom. Building on these foundations [129], [130], Borković et al. [131] presented the first truly GE isogeometric Bernoulli-Euler beam formulation, satisfying both objectivity and path-independence. This was followed by a formulation based on the Frenet-Serret frame that includes a rotation-free variant and an improved constitutive description for strongly curved beams [132]. These formulations were subsequently extended to account for interaction potentials between beams [133], [134], [135].

6. CONCLUSION

This review has outlined the mathematical and computational foundations required for reliable beam analysis under large displacements and finite rotations. The non-Euclidean structure of rotations on $SO(3)$ was shown to be central to both continuum modeling and finite element discretization, since the choice of rotation parametrization, interpolation, and update strategy directly affects objectivity, path independence, and numerical robustness. By revisiting classical shear-rigid and shear-deformable beam theories and tracing the development of geometrically exact (GE) formulations, the review distinguished the two main kinematic lines used in practice: GE Simo-Reissner beams and GE Kirchhoff-Love beams.

In finite element discretizations, numerical artifacts such as spurious strains and locking often originate from the treatment of rotational variables rather than from the continuum theory itself, motivating manifold-consistent interpolation and mixed formulations. The review focuses primarily on the kinematic and discretization aspects of GE beam formulations, while dynamics, stability, contact, and advanced material behavior are addressed selectively to highlight key extensions and representative contributions. Recent isogeometric work further indicates that spline-based discretizations can achieve high accuracy and robust performance, especially when combined with mixed or collocation schemes.

The GE beam formulation is a mechanical model that provides excellent balance between accuracy and efficiency. Although thoroughly investigated, it is still an active research topic. A particularly underexplored area concerns the isogeometric analysis of strongly curved Simo-Reissner beams. While the effect of strong curvature has been addressed for geometrically exact Kirchhoff-Love beams [126], [127], [129], [131], [132], no equivalent formulation currently exists for the Simo-Reissner model. Most existing formulations for arbitrarily curved beams still rely on simplified beam metrics and neglect higher-order curvature effects across the cross-section, which reduces accuracy when initial curvature becomes large.

Looking ahead, further advances are expected in the incorporation of advanced material models, contact algorithms, and efficient dynamic solvers within geometrically exact and isogeometric frameworks, where the consistent treatment of finite rotations remains a challenge.

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