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ISOGEOMETRIC – BASED DYNAMIC ANALYSIS OF BERNOULLI – EULER CURVED BEAM SUBJECTED TO MOVING LOAD

Abstract:

In this paper dynamic analysis of a curved Bernoulli – Euler beam subjected to a moving load is presented. Moving load is modelled as a single force with constant magnitude and direction, which moves along its trajectory. Plane curved Bernoulli – Euler beam element is formulated using isogeometric approach where both the displacement field and geometry of the beam are described using NURBS basis functions. Behavior of the beam element is defined and studied in the case of linear formulation where displacements and displacement gradients are assumed to be small. Validation of the proposed approach is presented for the plane curved beam subjected to moving load with constant velocity, magnitude and direction.

Keywords: isogeometric Bernoulli – Euler curved beam, moving load, linear analysis

ДИНАМИЧКА АНАЛИЗА РАВАНСКЕ ИЗОГЕОМЕТРИЈСКЕБЕРНУЛИ – ОЈЛЕРОВЕ КРИВЕ ГРЕДЕ ОПТЕРЕЋЕНЕ ПОКРЕТНИМ ОПТЕРЕЋЕЊЕМ ПРИМЕНОМ ИЗОГЕОМЕТРИЈСКОГ ПРИСТУПА

Сажетак:

У овом раду приказана је динамичка анализа криволинијског Бернули – Ојлеровог гредног носача оптерећеног покретним оптерећењем. Покретно оптерећење је дефинисано као концентрисана сила константног интензитета, правца и смера, која се креће по својој трајекторији. Раванска криволинијска греда је формулисана применом изогеометријског приступа где се поље померања описује истим функцијама као и геометрија конструкције, НУРБС функцијама. Анализа утицаја покретног оптерећења на конструкцију се врши у условима малих померања и градијената померања, тј. у условима линеарне анализе. Валидација приказаног приступа је дата на примеру раванске криволинијске греде која је оптерећена покретним оптерећењем константног интензитета, правца, смера и брзине кретања.

Кључне ријечи: изогеометријска Бернули – Ојлерова крива греда, покретно оптерећење, линеарна анализа

1. INTRODUCTION

Moving load generates dynamic response, which can be critical for bridges and cranes amongst others. This load, generated by the moving mass on the structure, is usually modelled as a gravitational force with constant magnitude and direction [1, 2]. Using this formulation the inertial part of moving mass is neglected which can be significant in some cases [3]. It is essential to define moving load trajectory and its position of the structure at each time. In linear dynamic analysis, the assumption that the moving load trajectory matches the undeformed structure geometry is valid and will be used in this formulation.

Curved structure geometry can be defined using CAD (*Computer Aided Design*) software packages, which are based on NURBS (*Non Uniform Rational B-Spline*) functions. These rational functions are used for their capability to exactly represent shapes of conical sections like circle, ellipse, parabola, hyperbola as well as free form curves. Consequently, the trajectory of the moving load can be obtained exactly using the NURBS basis functions.

Most of the software packages for structural analysis are based on finite element method (FEM). In order to apply FEM, physical domain of a structure has to be discretized, forming mesh of finite elements. This discretization is obtained from the structure's geometrical model. If the analysis results are not accurate, finer mesh is required which is obtained from the geometrical model of the structure. Back and forward procedure between the structural geometry and analysis model can use great computational and time resources, which represent a disadvantage of the FEM.

In order to overcome this disadvantage, the isogeometric approach (IGA) has been developed by Hughes and his co-workers [4] where solution space is formed using the same basis functions - NURBS that are used for geometry description. The focus of IGA utilization is on curved structural elements. For several years great effort has been devoted to the formulation of a Bernoulli – Euler curved elements for static and dynamic analysis [5 – 7].

In this paper dynamic analysis of a curved plane Bernoulli - Euler beam subjected to a moving load is presented. The trajectory of a load matches the beam geometry, which has been defined using NURBS basis functions. This assumption is valid for linear dynamic analysis. Plane curved beam is defined using Bernoulli – Euler beam theory as presented in [7]. All necessary elements have been implemented in MATLAB [8] and used to calculate dynamic response. The results obtained using the presented formulation are compared with the results from the literature.

2. BASICS OF NURBS

Geometry of a plane curve $\mathbf{C}(\xi)$ can be represented using NURBS parametrization as:

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi)\,\mathbf{C}_i \tag{1}$$

where ξ represents the independent parameter, $R_{i,p}(\xi)$ is the i-th NURBS basis function of degree p, while $\mathbf{C}_i$ is the i-th control point defined in Cartesian coordinate system. As can be noticed, basis vectors are defined in parametric domain using so-called knot vector composed of non-decreasing sets (ξ_i) of coordinates in parametric domain, called knots. NURBS functions as rational functions are constructed from B – Spline functions as:

$$R_{i,p}(\xi)=\sum_{i=1}^{n} \frac{N_{i,p}(\xi)\cdot w_i}{\sum_{j=1}^{n} N_{j,p}(\xi)\cdot w_j} \tag{2}$$

where w_i is i-th function weight. B – Spline functions are polynomial functions obtained using Cox de Boor algorithm. For the case of zero degree the B – Spline functions are defined as:

$$N_{i,p}(\xi) = \begin{cases} 1, & \text{if } \xi \in [\xi_i,\xi_{i+1}[\\ 0, & \text{otherwise} \end{cases} \tag{3}$$

while for the polynomial degree greater than zero:

$$N_{i,p}(\xi) = \begin{cases} \dfrac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \dfrac{\xi_{i+p-1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi), & \text{if } \xi \in [\xi_i, \xi_{i+p+1}[\\ 0, & \text{otherwise} \end{cases} \quad (4)$$

B – Spline functions have a property of non – negativity, partition of unity, and with adequate choice of knot vector, interpolator property at the domain boundary. The properties of B – Spline functions are inherited for NURBS basis functions, which is important for beam formulation. In Figure 1 plane curve with an arbitrary shape defined using four control points and adequate NURBS basis functions has been presented. More about B – Spline and NURBS functions and their properties and utilization can be found in [9].

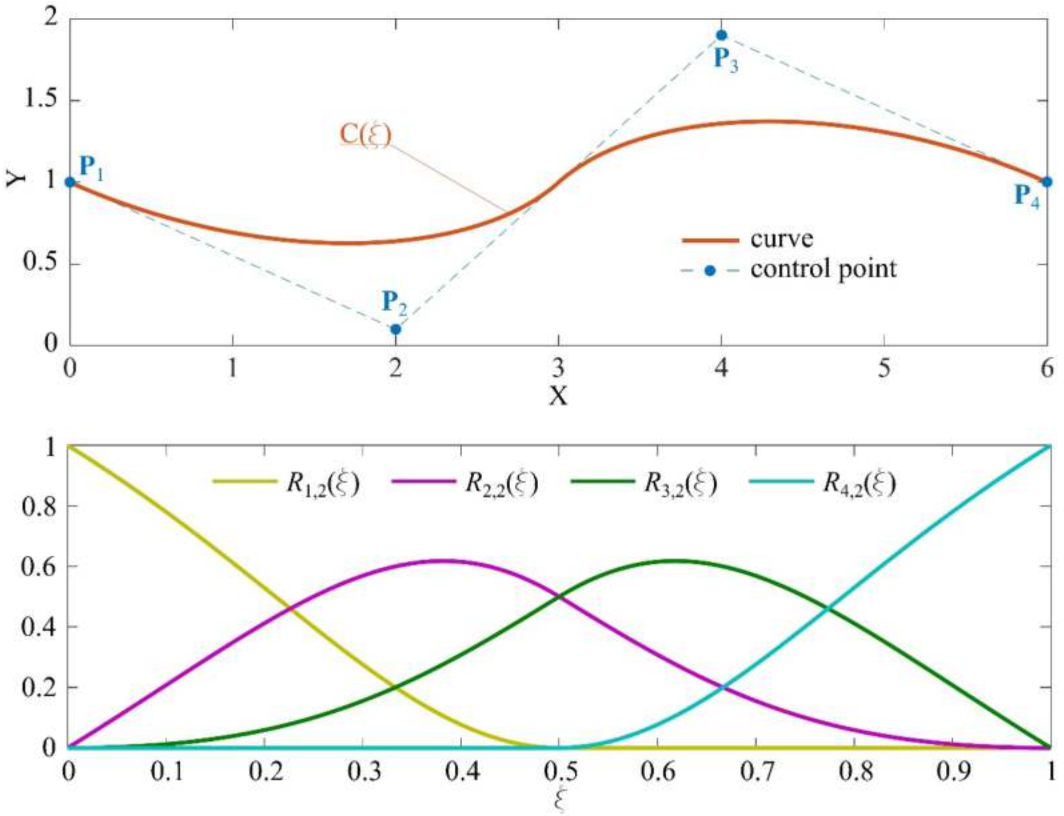

Figure 1. Plane curve and corresponding NURBS basis functions

3. BEAM GEOMETRY

Due to the assumption of undeformable beam's cross section, all beam quantities are defined at beam's centerline. Centerline of curved beam is curve line which can be parametrized using NURBS parametrization as:

$$\mathbf{r}(\xi) = \sum_{i=1}^{n} \mathrm{R}_{i,p}(\xi)\, \mathbf{r}_i \quad (5)$$

where $\mathbf{r}(\xi)$ is the position vector of beam's centreline, while $\mathbf{r}_i$ is the i-th control point, Figure 2. Using well-known relations of differential geometry [10] the basis vectors of plane curve are defined as:

$$\mathbf{g}_1 = \mathbf{r}_{,1} = \frac{d\mathbf{r}}{d\xi} = \frac{d\mathbf{r}}{ds}\frac{ds}{d\xi} = \mathbf{t}\frac{ds}{d\xi} = \mathbf{t}\sqrt{g_{11}} \quad (6)$$

$$\mathbf{g}_2 = \frac{\mathbf{K}}{K} = \frac{1}{K}\frac{d\xi}{ds}\frac{d}{d\xi}\left(\frac{\mathbf{g}_1}{|\mathbf{g}_1|}\right) \tag{7}$$

where $\mathbf{g}_1$ vector is the general non-unit vector collinear to tangent vector $\mathbf{t}$, $\mathbf{g}_2$ is the normal vector perpendicular to tangent vector thus lies in beam's cross section, $\mathbf{K}$ is the curvature vector with its modulus K, while s represents the arc – length coordinate. Metric tensor of presented reference frame is obtained as:

$$g_{ij} = \begin{bmatrix} g_{11} & 0 \\ 0 & 1 \end{bmatrix},\ det\left(g_{ij}\right) = g_{11} = g \tag{8}$$

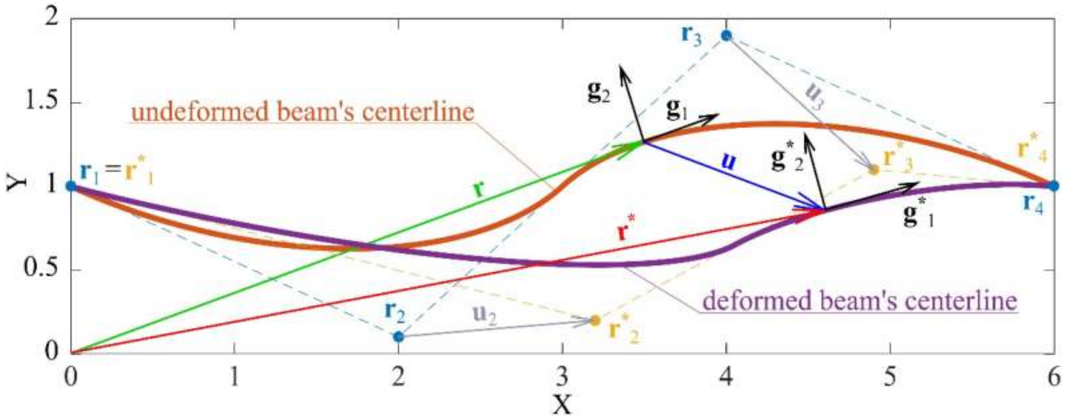

Figure 1. Undeformed and deformed beam's centerline

First derivative of basis vectors with respect to the parametric coordinate is obtained using Frenet – Serret relation as:

$$\begin{bmatrix} \mathbf{g}_{1,1} \\ \mathbf{g}_{2,1} \end{bmatrix} = \begin{bmatrix} \Gamma^1_{11} & gK \\ -K & 0 \end{bmatrix}\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} \tag{9}$$

where $\Gamma^1{}_{11}$ is Christoffel symbol of the second kind. Using the frame of reference, the position vector of an arbitrary point of beam is obtained as:

$$\tilde{\mathbf{r}} = \mathbf{r} + \eta\mathbf{g}_2 \tag{10}$$

where η represents the coordinate axis in the direction to the beam's cross section principle axis. From previous relation the basis vectors of an arbitrary point are:

$$\tilde{\mathbf{g}}_1 = (1\text{-}\ \eta K)\mathbf{g}_1 \tag{11}$$

$$\tilde{\mathbf{g}}_2 = \mathbf{g}_2 \tag{12}$$

As can be noticed, second basis vector is independent on the point position due to the assumption of rigid cross section. Metric tensor of an arbitrary point is obtained as:

$$\tilde{g}_{ij} = \begin{bmatrix} g_0\, g_{11} & 0 \\ 0 & 1 \end{bmatrix},\ g_0 = (1\text{-}\ \eta K)^2 \tag{13}$$

4. ISOGEOMETRIC BERNOULLI – EULER BEAM FORMULATION

Position vector of beam's centerline in deformed configuration is given as:

$$\mathbf{r}^* = \mathbf{r} + \mathbf{u} \tag{14}$$

where $\mathbf{u}$ represents the displacement vector of beam's centreline, Figure 2. If both undeformed and deformed beam configurations are parametrized using the same parametrization, then the displacement vector is defined as:

$$\mathbf{u}(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi)\, \mathbf{u}_i = \sum_{i=1}^{n} R_{i,p}(\xi)\, u_i^m \mathbf{i}_m \tag{15}$$

Eq. (15) represents the main property of the isogeometric approach where geometry and solution space are defined using the same basis functions.

Using convective system of reference, the deformation of the beam is contained in the deformation of beam's basis vectors as:

$$\mathbf{g}_m^* = \mathbf{g}_m + \mathbf{u}_m \tag{16}$$

In addition, displacement field of an arbitrary point is given as:

$$\tilde{\mathbf{u}} = \mathbf{u} + \eta \mathbf{u}_2 \tag{17}$$

Correspondingly, acceleration and displacement variations of an arbitrary beam point are given as:

$$\tilde{\mathbf{a}} = (\ddot{\tilde{\mathbf{u}}}) = \ddot{\mathbf{u}} + \eta \ddot{\mathbf{u}}_2 \tag{18}$$

$$\delta\tilde{\mathbf{u}} = \delta\mathbf{u} + \eta\, \delta\mathbf{u}_2 \tag{19}$$

As mentioned before, the beam formulation is given in the convective system of reference. Thus, the axial deformation term of the deformation tensor is obtained as:

$$\tilde{\varepsilon}_{11} = \frac{1}{2}\left(\tilde{g}_{11}^* - \tilde{g}_{11}\right) = g_0[(1+\eta K)\varepsilon_{11} - \eta\kappa] \tag{20}$$

where terms ε_{11} and κ represent respectively the strain deformation of the beam's centreline and bending deformation about the axis $\mathbf{g}_2$:

$$\varepsilon_{11} = \frac{1}{2}\left(g_{11}^* - g_{11}\right) \tag{21}$$

$$\kappa = \overline{K}^* - \overline{K} = \mathbf{g}_2 \cdot \left(\mathbf{u}_{1,1} - \Gamma_{11}^1 \mathbf{u}_{,1}\right) \tag{22}$$

In this paper, generalized Hook's law is used in order to define relation between stress and deformation terms:

$$\tilde{\sigma}^{ij} = \frac{E}{1+v}\left(\tilde{g}^{ik}\tilde{g}^{jl}\tilde{\varepsilon}_{kl} + v\tilde{g}^{ij}\tilde{g}^{11}\tilde{\varepsilon}_{11}\right) \tag{23}$$

where E is Young's modulus, while v represents the Poisson's coefficient. In order to obtain equations of motion, the principle of virtual work is used:

$$\int_V \rho \tilde{\mathbf{a}} \cdot \delta\mathbf{u}\, dV + \int_V \mathbf{S} : \delta\mathbf{E}\, dV = \int_l \mathbf{f} \cdot \delta\mathbf{u}\, dx \tag{24}$$

where ρ is the mass density, **S** is the second Piola-Kirchoff stress tensor, $\delta\mathbf{E}$ is variation of the Green-Lagrangian strain tensor, while **f** is the external load. Applying Eqs. (18), (19), (20) and (23), the governing equation of moving load problem on curved beam is obtained:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q} \tag{25}$$

where **M** is the mass matrix, **K** is the stiffness matrix, **Q** is the vector of equivalent control forces, while **q** is the displacement vector of the control points. In order to solve Eq. (25), numerical step by step integration has been applied based on the finite difference method. Also, for calculation of mass and stiffness matrices, given in [7], as well as vector of equivalent forces, reduced numerical integration [11] has been applied and implemented in original MATLAB [8] code.

5. MOVING LOAD

Moving load is a spatially varying load, which generates dynamic response of a structure. This load can be modelled as a single force with constant magnitude ($\mathbf{f}_0$) and direction, which moves along a beam with velocity V_ξ:

$$\mathbf{f}(t) = \mathbf{f}_0 \cdot \delta(\xi - V_\xi t),\ V_\xi = \frac{d\xi}{dt} = \frac{V}{\sqrt{g}} \tag{26}$$

where V_ξ and V are the velocity magnitudes given in NURBS and arc - length parametrizations, respectively. Point moving load transformed with respect to the integration points is presented in Figure 3.

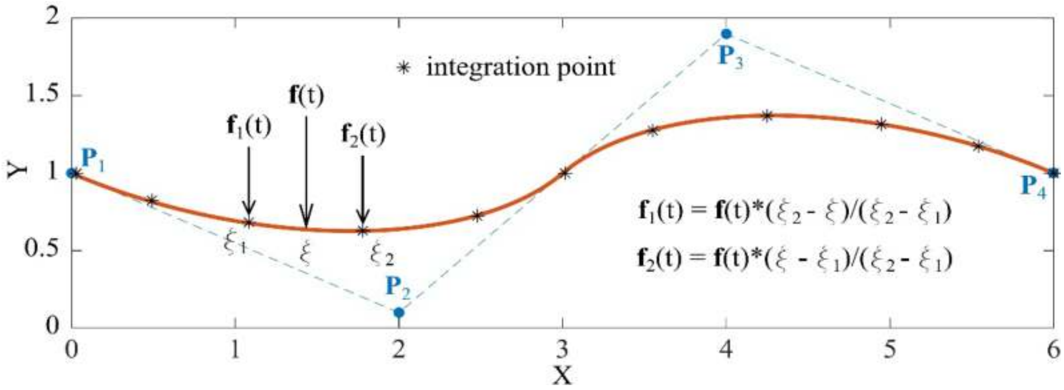

Figure 2. Moving load distribution on integration points

6. NUMERICAL EXAMPLE

To illustrate and validate the proposed method, dynamic analysis of simply supported plane curved beam subjected to the moving load is carried out. Geometry, material properties and load of the beam are given in Figure 4. Beam geometry is generated using the following control points:

$$\mathbf{r}^T = \begin{bmatrix} 0 & 5 & 10 \\ 0 & 5\cdot\tan(\pi/6) & 0 \end{bmatrix} \tag{27}$$

and NURBS basis functions of degree 2 constructed using knot vector $\boldsymbol{\xi}^T = [0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1]$ and weights $\mathbf{w}^T = [1\ \ \sin(\pi/3)\ \ 1]$. Applied force has magnitude of 0.106 *kN* and moves along the beam with velocity $V = 8.1$ *m/s*.

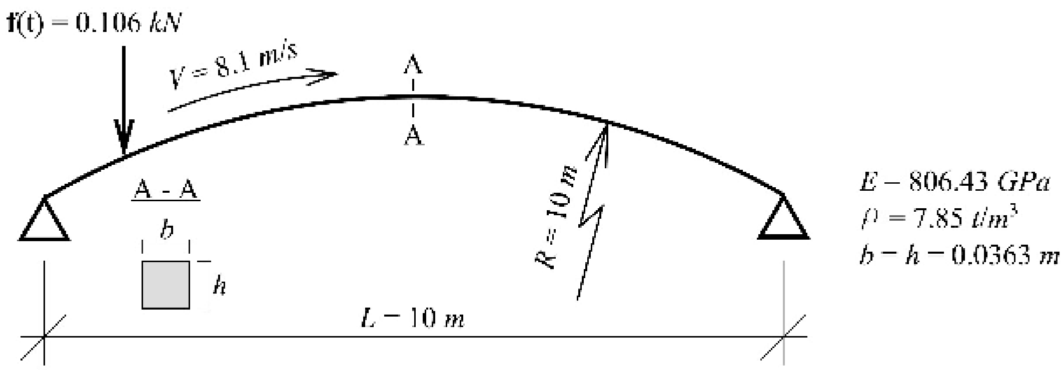

Figure 3. Simply supported curved beam subjected to the moving load with constant velocity and magnitude

The convergence of the presented approach has been investigated using the h – refinement, which is achieved by knot insertion in the parametric domain. By applying this refinement procedure, the geometry of structure remains unchanged while the number of degrees of freedom (DOF) increases.

In this example, four beam models are analyzed with different number of DOFs: Model 1 (10 DOFs), Model 2 (22 DOFs), Model 3 (42 DOFs), Model 4 (82 DOFs). In Figure 5, time history of the displacement at the position of the moving load is presented. The results converged in Model 3 with 42 DOFs. However, some discrepancies have been noticed in comparison with the results reported in [3]. These discrepancies occurred due to the applied beam model based on Timoshenko theory.

7. CONCLUSIONS

In this paper the dynamic analysis of a curved plane Bernoulli – Euler beam subjected to a moving load is presented. The moving load is modelled as a point force with constant magnitude and direction, while the curved beam is modelled using the isogeometric approach. It is assumed that the moving load trajectory matches the shape of the undeformed beam. In order to validate presented formulation the numerical example of a moving load on a curved plane beam has been carried out. Good agreement between the results obtained using the presented approach and the results from the literature has been shown. For future research, the dynamic analysis of a plane curved beam subjected to a moving mass will be investigated. In addition, the influence of moving load and mass can be extended to the case of spatial curved beam element.

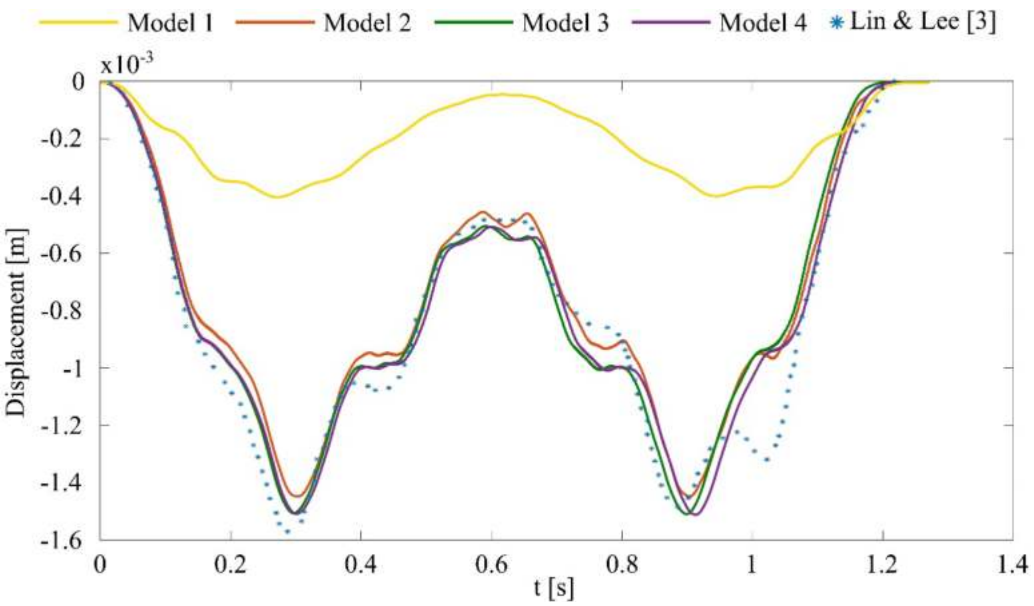

Figure 4. Vertical displacement of beam at the position of the moving load

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