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REVIEW OF BASIC CONCEPTS OF CONTACT MECHANICS OF FRICTIONLESS CONTACT WITH EMPHASIS ON PENALTY METHOD

Abstract:

Problems involving contact are of great importance in industry related to mechanical and civil engineering, but also in biomechanics and other applications. The contact interaction between surfaces and different bodies like in a bolted splice connection joint or area in which a tire interacts with the soil is not known *a priori*, leading to a nonlinear boundary value problem. Due to the rapid improvement of modern computer technology, one can today apply the tools of computational mechanics to analyze contact problems with required accuracy, depending on design requirements. However, even now most of the standard finite element software is not fully capable of solving contact problems, including friction, with robust algorithms. The aim of this paper is to present some basic concepts of Contact Mechanich.

Keywords: Contact Mechanics, penalty method, frictionless contact

ПРЕГЛЕД ОСНОВНИХ КОНЦЕПАТА КОНТАКТНЕ МЕХАНИКЕ ЗА КОНТАКТ БЕЗ ТРЕЊА СА НАГЛАСКОМ НА PENALTY МЕТОД

Сажетак:

Проблеми који укључују контакт од велике су важности у машинској и грађевинској индустрији али такође и у биомеханици, металургији и другим гранамаобластима. Контактна интеракција између различитих тијела и површина, као на примјер вијчане везе монтажног наставка или интеракције аутомобилске гуме и подлоге, нису познате *a priori*, што доводи до нелинеарног граничног проблемаграничнихуслова. Брзим напретком модерне рачунарске технологије, данас је могуће примјенити алате нумеричке механике за анализу контактних проблема у оквиру ограниченетражене тачности, зависно од захтјева пројекта. Међутим, већина стандардних софтвера са коначним елементима још увијек није у могућности да ријеши контактне проблеме укључујући трење. Циљ овог рада јесте представити основне концепте контактне механике.

Кључне ријечи: Контактна механика, пеналту-метода, контакти без трења

1. INTRODUCTION

From mechanical point of view contact is included in all forms of interaction between two bodies which come in touch with one to the other. Contact between solid bodies enables transition of force, temperature and electrical charge from one body to the other therefore we can conclude that describing a contact is one highly demandable and robust mathematical problem that is still far from finished [1]. Contact mechanics is included in wide range of problems that we face, like for an example a vehicle collision with load bearing column, sheet metal forming, car impact with deformable barrier, steel box girders plastic deformation with self-contact and also study of land slides, avalanches and etc. Some of above-mentioned problems are shown in the Fig.1.
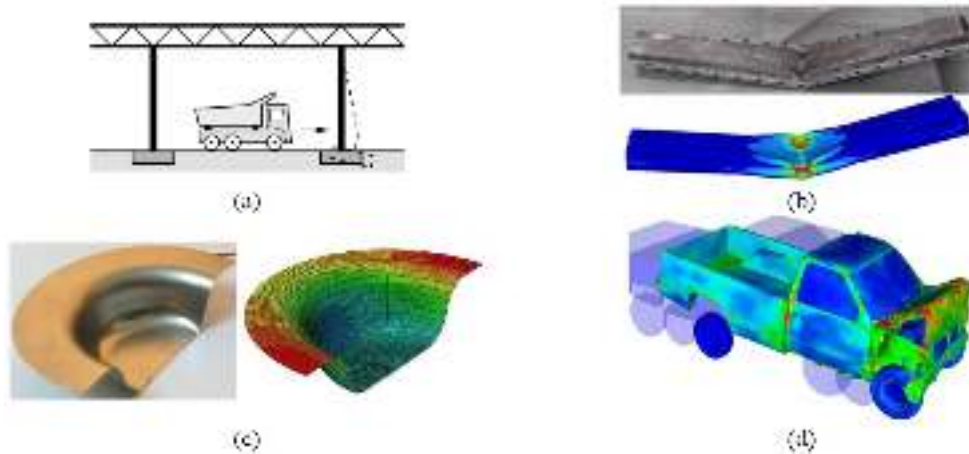

Figure 1 Vehicle collision with bearing column (a), steel box girder plastic deformation with self-contact (b), sheet metal forming (c), car impact in barrier (d)

Mechanics of deformable body problems that include contact of two bodies can be seen as a special group of problems, because contact forms as a consequence of two separate continuous surface interface. Limitations that occur on the place of interface cannot be seen and replaced with an usual boundary conditions for both contact surfaces used in mechanic of deformable solid body. At the same time, neither the contact zone could be considered as continual system. By idealization of contact, we can observe area of contact surfaces as: a zero thickness layer that can transfer only the pressure forces which acts perpendicular to the surface of interface Fig. 2a, while the occurrence of tensile forces leads to vanishing the contact interaction of bodies and comes to separation Fig. 2b. In the case of contact with classic Coulomb's friction, in the case without slipping – stick state both surfaces are fixed for each other Fig. 2c [3].
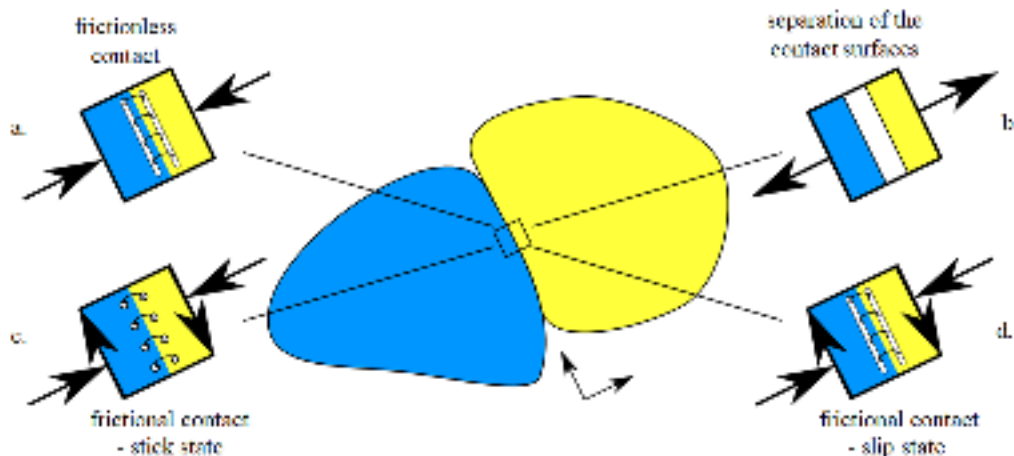

Figure 2 An analogy for contact interface: a – frictionless contact sustains compressive stress, b – any stretching leads to vanishing of contact interface, c – frictional contact interface can transfer shear stress; d – in Coulomb's friction law in stick state there is no relative sliding up to reaching critical shear stress. experimental and numerical cantilever beam with bolted splice connection joint [3]

Such contact can be seen as a restraint where the normal and shear stress occur between contact surfaces. Slipping occurs after reaching the critical shear stress between contact surfaces – slip state, Fig. 2d.

For the sake of complexity of describing the contact interaction, in this paper, we will consider a simple numerical example of frictionless normal contact for a linear elastic material. Contacts without friction that only consider normal stress in calculation and are based on Hertz-s theory are defined as normal contacts where friction i.e. tangential force is neglected [4] [5].

We will see that in the case of contact with small deformations, by the linear relationship of the stress-strain curve and in case of frictionless contact, we have a nonlinear response again due to the change of contact boundary conditions. Given the always nonlinear contact behavior and non-smooth contact law for the normal contact pressure, it is necessary to apply some of numerical methods so to overcome this problem. In this paper, Penalty method which represents one of usually applied numerical methods to solve contact problems will be considered.

2. HERTZIAN CONTACT AND BASIC CONCEPTS OF ANALYTICAL CONTACT

As aforementioned in later work, we will deal with normal contact problems that consider two bodies brought in contact by forces perpendicular to their plains. The simplest contact problem is contact between orthogonal parallelepiped and smooth rigid plane without friction Fig. 3.

Figure 3 Contact between an elastic parallelepiped and rigid plane

When the body is pressed against a plain, deformation occurs, and value d called the "penetration depth" which represents depth of penetration body to plane if plane is not rigid. After the uniaxial stress state is adopted, resulting force of contact surfaces is equal:

$$F = EA\frac{d}{l} \tag{1}$$

From aforementioned example comes the direct connection between penetration depth d and module of elasticity E [4]. Depending on depth of penetration and resulting force, the hardness of material can be determined, which besides the roughness has the crucial part in contacts with friction. Indenting the circular body with constant diameter D in to deformable surface, strength of materials by Brinell is got, from force F_N and area under circular body A ratio, Fig. 4.
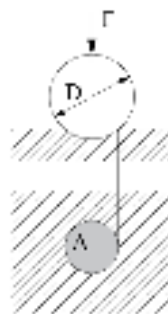

Figure 4 Brinell hardness test

$$\sigma_0 = \frac{F_N}{A} \tag{2}$$

Importance of penetration depth comes to expression while describing contact surface discretization, defining and restriction of the finite elements penetration one into other, search and detection of finite elements which will get in contact and defining of normal and tangential gap function.

2.2. Contact between a rigid cylinder and elastic half-space

With further example the basic principles of contact mechanics will be explained. We will observe the contact of rigid cylindrical body and elastic half-space. If it is adopted that stress acts over limited plane with length D Fig. 5, then both the deformation and stress on plane D are the same magnitude in all three directions over volume for length D. To determine the fundamental settings of contact problem we can adopt the presumption that deformation is constant over the aforementioned volume with dimensions D, and that only this volume is deformed. From elastic half-space deformation point of view, this is an approximate presumption of real stress and deformation distribution inside the continuum. Nevertheless, this method can be obtained enough good results and a qualitative connection of contact force and penetration depth, respectively the radius of the contact plane.

We apply the defined rules on the aforementioned example with stiff cylindrical body [4] [5]. When the cylinder radius is equal 2a, then volume in all three directions for length 2a is heavily deformed similar like the deformed volume below indenter from the previous example. If the volume in the place of contact pressure is deformed through depth d independent of overall deformation 2a, deformation can be written as $\varepsilon \approx d / 2a$. It follows that the stress is $\varepsilon \approx d / 2a$ and the force is $F \approx \sigma (2a)^2 \approx 2Eda$. Contact force is proportional to depth of penetration d and the radius of contact interaction a. By comparing approximate with exact solution, only 10% of difference in result is got. Exact solution is written by expression:

$$F = 2E^* da, \quad \text{where} \quad E^* = \frac{E}{1 - v^2} \quad (3)$$
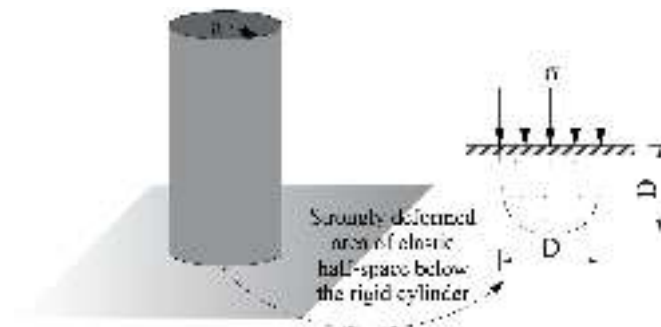

Figure 5 Contact between a rigid cylindrical indenter and elastic half-space with detail of the strongly deformed area of elastic half-space below of indenter

 Based on described example, approximate method besides qualitative can be used for quantitative evaluation. From the equation (3), it is seen that the penetration depth is proportional to normal force, and $2E^* a = k$ represents rigidity of contact plains.

2.3. Hertz - contact theory

Hertz contact belongs to the classical problem of description of normal contact between rigid sphere and elastic half space, which is used as a base for more complex models, Fig. 6 [4].
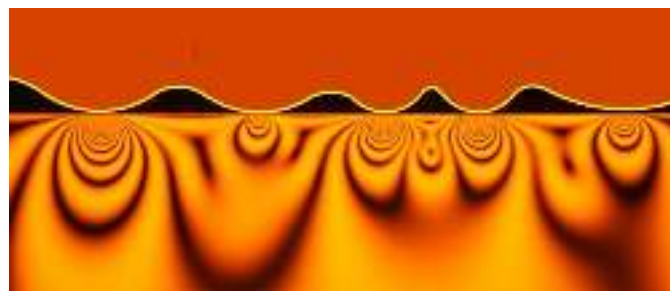

Figure 6 Visualization of Hertzian distribution of stress for a contact of a spherically rough rigid surface with an elastic plate

Elastic medium is seen as infinite half space wich only limitation is infinite plane. Under influence of force which acts upon free surface the elastic medium will deform. As was aforementioned only normal contact is considered so we will observe plane xy on the surface of elastic medium upon which acts force F in positive direction Z. In particular, we consider the displacements of the free surface, which means that we have defined as Z = 0 from where follows $r = \sqrt{x^2 + y^2}$. The final expressions for displacement are defined as:

$$u_x = -\frac{(1+v)(1-2v)}{2\pi E}\frac{x}{r^2}F_z \tag{4}$$

$$u_y = -\frac{(1+v)(1-2v)}{2\pi E}\frac{y}{r^2}F_z \tag{5}$$

$$u_z = \frac{(1-2v)}{\pi E}\frac{x}{r}F_z \tag{6}$$

For contact problems without friction, where only normal force is observed, that is, normal pressure $p_{(x,y)}$, only Z component is of interest in approximation of half-space (7). Given that Hertz-s theory is based on contact of spherical peaks, final expression for translation in y direction will be written through cylindrical coordinates (8) in regard of pressure $p(s, \phi)$ with formerly adopted coordinate Z = 0 [5].

$$u_z = \frac{1}{\pi E^*}\iint p(\xi,\eta)\frac{d\xi\,d\eta}{r}, \qquad r = \sqrt{(x-\xi)^2 + (y-\eta)^2} \tag{7}$$

$$u_z = \frac{1}{\pi E^*}\iint p(s,\phi)\;ds\,d\phi \tag{8}$$

For vertical displacement of surface points of elastic half-space it is presumed that distribution of pressure according to $p = p_0(1 - r^2/a^2)^n$ is applied on circular surface with radius a by which the pressure and contact is exerted with free surface of half space. When n = 1/2 we get Hertzian pressure distribution.

$$p = p_0\left(1 - \frac{r^2}{a^2}\right)^{\frac{1}{2}} \tag{9}$$

The resulting vertical displacement and total force follows as:

$$u_z = \frac{\pi p_0}{4E^* a}\left(2a^2 - r^2\right) \tag{10}$$

$$F = \int_0^a p(r)\,2\pi r dr = \frac{2}{3}p_0\pi a^2 \tag{11}$$

 For further defining of Hertz-s problem, contact of rigid sphere with radius R and elastic half space will be observed Fig. 7. The displacement of points on the surface contact area between rigid sphere and flat deformable base is equal:

$$u_z = d - \frac{r^2}{2R} \tag{12}$$

The schematic representation of Hertz-s contact between a rigid sphere and deformable surfaces is reminiscent of the Brinell hardness test previously mentioned [4], where the depth of sphere penetration d into a plane, and later also the radius of contact surface a is found as main parameters.

Figure 7 A rigid sphere in contact with an elastic half-space

Based on adopted assumptions and obtained expressions, we can write equations that define the Hertz contact of a rigid sphere and a deformable plane.

$$a^2 = Rd \qquad \text{contact radius} \tag{13}$$

$$p_0 = \frac{2}{\pi} E^* \left(\frac{d}{R} \right)^{1/2} \quad \text{maximum contact pressure} \tag{14}$$

$$F = \frac{4}{3} E^* R^{1/2} d^{3/2} \quad \text{normal force} \tag{15}$$

2.4. Contact between rough surfaces

Once we have defined the Hertz contact between sphere and plane, we can consider the case of the interaction of rough surfaces without friction, where between the contact surfaces due to roughness, the contact is unevenly spread at several different points. The total contact area (actual) between two rough bodies is much smaller than the (visible) surface that appears to be in contact. The size of the actual contact surface is an important factor, especially in contacts with friction involved, where it significantly affects energy dissipation and damping due to contact interaction. We can say that the cause of friction is actually the breaking, elastic and plastic deformation of the contact micro asperities. In this section, we study the interaction of rough surfaces and rigid plane.

Describing the interaction of rough contact surfaces it gets further complicated when we consider that the roughness of the real surface and thus the actual contact surface are stochastically distributed. The simplest method and most basic model for describing irregular surfaces is the Greenwood model (J. A. Greenwood and J. B. P. Williamson), [4] [5]. Greenwood's model assumes that the roughness or contact points - asperities have the same radius according to Hertz's theory and that the height of the asperities (peaks) is stochastically distributed around some mean value, Fig. 8.
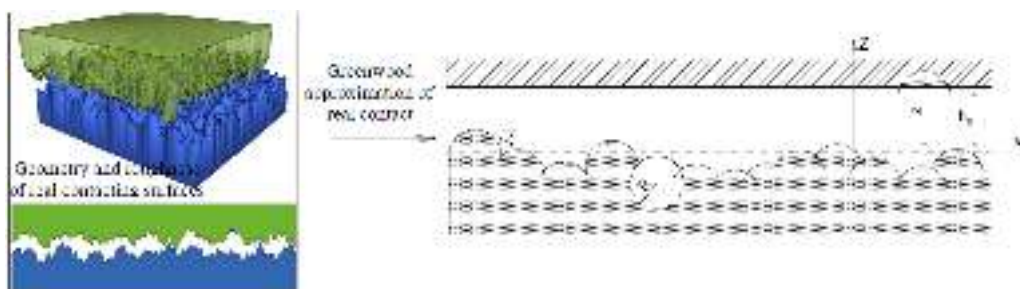

Figure 8 Greenwood approximation of real contact

As we can see, the question arises as to whether and when the bodies are in contact and which peak will first touch the body (rigid plane), Fig. 8. If we accept the assumption that the contact peaks are far enough from each other, then the deformation of each peak can be observed separately. Therefore, the position of the contact micro peaks and the configuration of the rough surfaces under the aforementioned assumption are not of great importance. Based on the assumptions we have made so far. We can conclude that if the contact peaks are sufficiently far from each other, then only the high distribution of the peaks is important.

We will define the probability that a given density of asperity has a maximum peak z, as a density function $\Phi(z)$. That means that the probability that the peak has a maximum value at the interval $[z, z+dz]$ is equal to $\Phi(z)dz$. If the total number of peaks in contact is N_0, then the total number

of peaks in the interval [z,z+dz] is equal to $N_0\Phi(z)dz$ [4]. For many surfaces depending on the material, it can be assumed normal distribution of the height of the peaks.

$$\Phi(z) = \left(\frac{1}{2\pi l^2}\right)^{1/2} e^{-\frac{z^2}{2l^2}} \tag{16}$$

Where l represents the root mean square (RMS) or contact surface roughness, the peaks height distribution:

$$l = \sqrt{\overline{z^2}} \tag{17}$$

Based on the aforementioned, from now we consider the contact between a rigid surface and an elastic body with statistically distributed roughness (peaks) at a distance h_0 from the coordinate origin of the z-axis, Fig. 8. The h_0 represents distance of the solid surface to the mean line of rough surface representing the mean value of the heights of all contact peaks. Assuming that there is no elastic interaction of the peaks between each other, all peaks with height $z > h_0$ are in contact with the rigid plate. The penetration depth of the peak with height z is $d = z - h_0$. It is more accurate to say that all peaks exceeding the distance h_0 will be deformed (crushed), and will be in contact with a rigid surface in small circular surfaces of radius a. Each individual contact peak was defined according to the Hertz rule $a^2 = Rd$. Hence based on already defined Hertz contact (13 to 15), we get an expression for the contact surface of one peak:

$$\Delta A = \pi a^2 = \pi dR = \pi(z - h_0) R \tag{18}$$

and for the force of the single peak we get:

$$\Delta F = \frac{4}{3} E^* R^{1/2} d^{3/2} = \frac{4}{3} E^* R^{1/2} (z - h_0)^{3/2} \tag{19}$$

By integrating the intervals of all contact peaks, we obtained the total number of points (peaks) that come in contact with the rigid plane N, the total contact area A and the total normal force F_N, which means that the integration must be carried out for all heights from $z = h_0$ to infinity [4] [5]:

$$N = \int_{h_0}^{\infty} N_0 \Phi(z) dz \tag{20}$$

$$A = \int_{h_0}^{\infty} N_0 \Phi(z) \pi R (z - h_0) dz \tag{21}$$

$$F_N = \int_{h_0}^{\infty} N_0 \Phi(z) E^* R^{1/2} (z - h_0)^{3/2} dz \tag{22}$$

The total number, total surface area, and total contact force increase exponentially as the bodies approach each other due to normal pressure (decreasing h_0). The average contact area of one peak can be obtained from the expression

$$\Delta A = \frac{A}{N} = \frac{\int_{h_0}^{\infty} N_0 \Phi(z) \pi R (z - h_0) dz}{\int_{h_0}^{\infty} N_0 \Phi(z) dz} \tag{23}$$

3. COMPUTATIONAL CONTACT MECHANICS CONCEPTS OF NORMAL CONTACT

Particularly demanding nonlinear problems to analyze is the contact between two or more bodies [1-3]. Contact problems can range from simple approximations of frictionless contact with small displacements to frictional contact with large displacements, large rotations and large strains.

3.1. Continuum mechanic formulations of contact

The nonlinearity of the analysis problem from now does not depend only on material and geometrical nonlinearity, which is usually considered for deformable bodies, but from contact conditions which are now included in the equation. Equation of balance in terms of current configuration expressed through Cauchy-s stress tensor is defined as [2]:

$$\int_{t_V}^{a} {}^t\tau_{ij}\delta_t e_{ij} \ d^t V = \int_{t_V}^{a} {}^t f_i^B \delta u_i \ d^t V + \int_{t_{S_f}}^{a} {}^t f_i^S \delta u_i^S \ d^t S \tag{24}$$

- ${}^t\tau_{ij}$: Cauchy stress tensor
- $\delta_t e_{ij}$: strain tensor corresponding to virtual displacements
- δu_i : components of virtual displacement vector imposed on configuration at time t
- ${}^t V$: volume at time t
- ${}^t f_i^B$: components of externally applied force per unit volume at time t
- ${}^t f_i^S$: components of externally applied surface tractions per unit surface area at time t
- ${}^t S_f$: surface at time t on which external tractions are applied
- $\delta u_i^S = \delta u_i$: components of virtual displacement vector
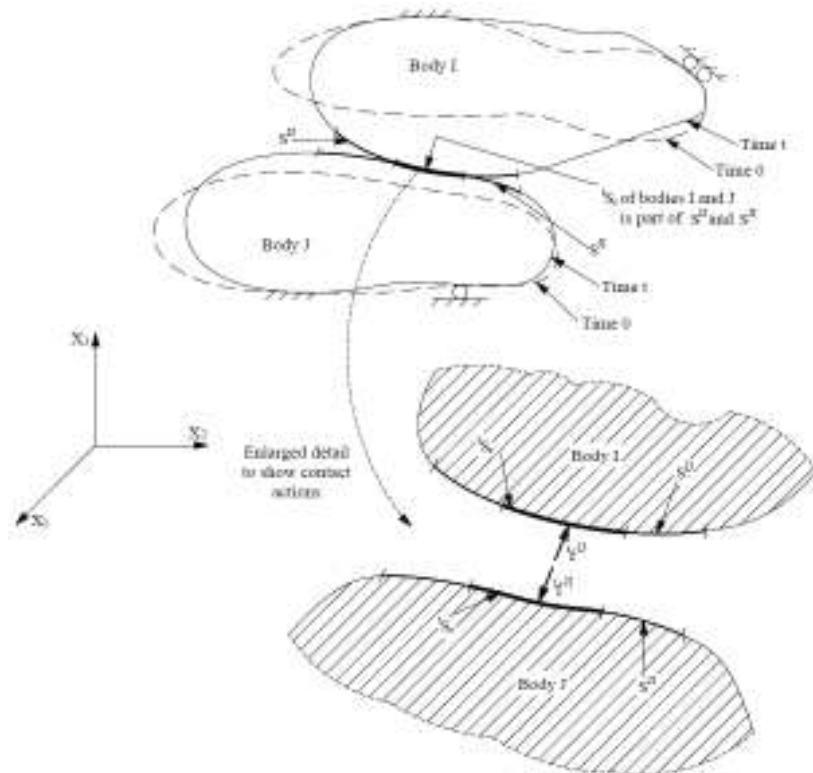

Figure 9 *Bodies in contact at time t*

Considering the contact problem, the equation of equilibrium for the N bodies in the contact on the right-hand side next to the expression of the external virtual work also contains the virtual work of the contact interaction (5). If L bodies are involved in the contacts $L = 1, \ldots, N$; where ${}^t S_c$ represents the total contact surface of each body, then the principle of virtual work for N number of bodies at time t is defined by the following expression [2]:

$$\sum_{L=1}^{N} \left\{ \int_{{}^t V} {}^t \tau_{ij} \, \delta_t e_{ij}^T \, d^t V \right\} = \sum_{L=1}^{N} \left\{ \int_{{}^t V} {}^t f_i^B \delta u_i \, d^t V + \int_{{}^t S_f} {}^t f_i^S \delta u_i^S \, d^t S \right\} + \sum_{L=1}^{N} \int_{{}^t S_c} {}^t f_i^c \delta u_i^c \, d^t S \tag{25}$$

Where part of a braces corresponds to the usual terms (4), while the last summation sign gives the force influence in a contact. As we can see contact force is represented as an exterior force. Components of this force are:

- ${}^t S_c$: complete contact area for each body L, L=1,..., N at the time t
- ${}^t f_i^c$: component of the contact traction act over the areas ${}^t S_c$
- ${}^t f_i^S$: components of the known externally applied tractions act over the surface ${}^t S_f$
- δu_i^c : components of the virtual displacement on the contact surface

Figure 9, shows two bodies in contact, which can be generalized to N bodies in contact. Two bodies are shown, the body I and body J, we notice that both bodies are constrained such that without contact no rigid body motions is possible. Let ${}^t f^{IJ}$ represent the contact force of body I due to contact with body J, so that from there it follows for body J that ${}^t f^{IJ} = - {}^t f^{JI}$. Now virtual work due to contact interaction can be represented as:

$$\int_{S^{IJ}} {}^t f_i^{IJ} \delta u_i^I \, dS^{IJ} + \int_{S^{JI}} {}^t f_i^{JI} \delta u_i^J \, dS^{JI} = \int_{S^{IJ}} {}^t f_i^{IJ} \delta u_i^{IJ} \, dS^{IJ} \tag{26}$$

where δu_i^I i δu_i^J are components of virtual displacement at the contact surface of bodies I and J.

$$\delta u_i^{IJ} = \delta u_i^I - \delta u_i^J \tag{27}$$

The surfaces S^{JI} i S^{IJ} denote the contact surfaces and are not necessarily to be the same size. The actual contact surface ${}^t S_c$ is actually a joint of the parts of the surfaces S^{JI} i S^{IJ} that come into contact at time t. Thus, we can represent the right side of equation (26) as a virtual work that produces a contact tract vector by virtual displacement on the contact surface [2].

By further developing of contact virtual work in the expression (25) and by application of Hertz-Signoriny-Morau condition, expression (25) transforms from usual formulation where solution which needs to satisfy equilibrium equation goes into inequality of equilibrium which further complicates defining of contact interaction of two bodies. For a detailed treatment of this subject the reader should consult the literature, e.g. [1-5]

3.2. Frictionless normal contact and defining of hertz-signorini moreau conditions

Consider a frictionless contact problem consisting of a concentrated mass m under a gravitational load supported by a stiffness spring k. The deflection of mass m is restricted by the rigid plane Fig.10. The potential energy of this system is:

$$\Pi_{(u)} = \frac{1}{2} k u^2 - mgu \tag{28}$$

The first term in the equation is the elastic potential of the spring and the second term represents the potential energy of mass in the gravitational field [1].
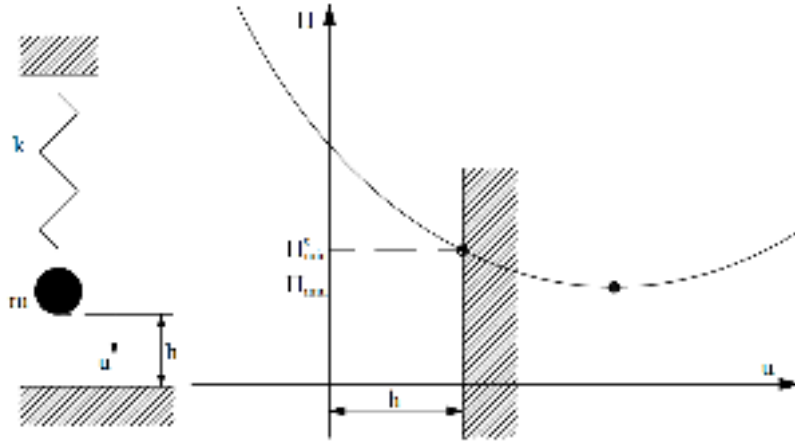
Figure 10. *Point mass supported by spring (a), energy of the mass spring system (b*

In the case that no rigid plane restriction for displacement u, we could define the extremum by the variational principle by introducing the variation of u as δu.

$$\Pi_{(u+\delta u)} = \frac{1}{2}k(u+\delta u)^2 - mg(u+\delta u)$$

$$= \frac{1}{2}ku^2 + ku\delta u + \frac{1}{2}k\delta u^2 - mgu - mg\delta u$$

$$= \frac{1}{2}ku^2 - mgu + \delta u(ku - mg) + \frac{1}{2}k\delta u^2 \tag{29}$$

$$= \Pi_{(u)} + \delta\Pi_{(u)} + \delta^2\Pi_{(u)}$$

Where we get the first variation as:

$$\delta\Pi_{(u)} = ku\delta u - mg\delta u \tag{30}$$

Where $\delta\Pi_{(u)} = 0 \; \forall \; \delta u$ represents the stationarity of the potential which is always true for the equilibrium state of the elastic system, from which it follows that the value for the first variation is a simple form of the virtual displacement principle.

The second variation of Π yields to:

$$\delta^2\Pi = k, \tag{31}$$

it follows that the system has a minimum defined by the expression

$$c(u) = h - u \geq 0 \tag{32}$$

Where penetration is excluded in the inequality of contact boundary condition, the body can only be in contact or separated from another body. For $c(u) > 0$ one has a gap between point mass m and the rigid plane, in the case that $c(u) = 0$ then we consider the gap is closed and contact is active [1].

We can observe that now the variation of δu is limited by the contact surface, from which it follows that it must be defined as $\delta u \leq 0$. Which means that the virtual displacement must meet the boundary condition of the contact, so that the variation can only be directed upwards. Based on all of the above, we define the expression for the first variation (30) as variational inequality:

$$ku\delta u - mg\delta u \geq 0 \tag{33}$$

In the inequality, the greater sign follows when the force mg is greater than the force in the spring kh and then the contact between the mass and the rigid surface is realized, which means that the

variation is $\delta u < 0$. Due to the above limitations, the minimum in (28) defined as Π_{min} Fig. 10, can no longer be taken as the minimum value of the system. The new minimum value of the system is now limited by the rigid plane or the position of the second contact body Π_{min}^c which we call the minimum energy in the allowed solution space Fig. 10.

The variation respectively virtual displacement δu can be represented by the difference between test function $v = \delta u + u$ and displacement u as $\delta u = v - u$.

$$ku(v-h) \geq mg(v-h) \tag{34}$$

The test function should fulfill the condition $v - u \leq 0$ on the contact surface, as well as the solution for u. The expression (30) written with the test function looks like:

$$ku(v-u) - mg(v-u) = 0 \tag{35}$$

When the mg > ku point mass comes into contact with a rigid plate and we have $v - h \leq 0$ the following is the inequality of the boundary condition of the contact:

$$ku(v-h) \geq mg(v-h) \tag{36}$$

In both cases, inequality (32) defines a displacement constraint, which leads to the variational inequality that characterizes the solution of u. The variational inequalities we define cannot be directly applied to solve the contact problem. The most commonly used methods are the Lagrange Multiplier method or the Penalty method, which will be explained in the following sections.
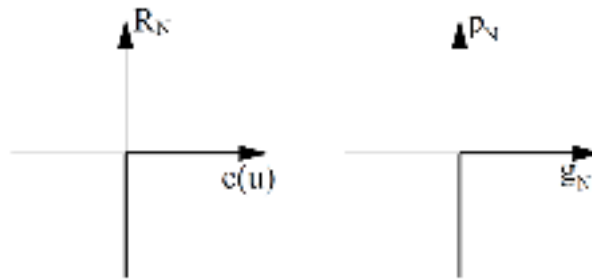

Figure 11. *Reaction force R_N versus gap $c(u)$ and contact force $p(n)$ versus normal gap $g(n)$*

Once the point mass m touches a rigid plane the reaction force R_N appears, we can say that we have an active constraint. We assume that only pressure can occur in contact, and we accept it as negative, while tensile force respectively adhesion is excluded. The reaction force that occurs in contact leads to the following condition:

$$R_N \leq 0 \tag{37}$$

meaning that either we have pressure and an active state in contact interaction $R_N < 0$ or an inactive reaction force which means that contact is not accomplished $R_N = 0$. Based on all of the above, we can define two cases for a contact problem where motion is restricted by condition (32).

1. The spring stiffness is sufficiently large that the mass cannot touch the rigid plane. In this case, the following conditions are valid:

$$c(u) > 0 \quad i \quad R_N = 0 \tag{38}$$

The ratio of mass to stiffness in the system is such that the mass comes in to contact with the rigid plane. In this case, the following condition applies:

$$c(u) = 0 \quad i \quad R_N < 0 \tag{39}$$

From the preceding two cases, we can form the Hertz - Signorini - Moreau condition

$$c(u) \geq 0, \quad R_N \leq 0, \quad R_N c(u) = 0 \qquad (40)$$

$$g_n \geq 0, \quad p_N \leq 0, \quad p_N g_n = 0 \qquad (41)$$

As we can see by comparing the derived basic conditions (40) and the final Hertz - Signorini - Moreau condition [1] of the deformable body (41), the simplified method of determining the contact boundary conditions on the based on example of mass and spring is the basis for the final Hertz - Signorini - Moreau condition for frictionless and no penetration contacts.

The gap function g_n in the simplified model defined as $c(u)$ (32) represents the dependence of spring stiffness, displacement, and mass. The stress vector component of the contact interaction p_N - contact pressure, is represented as the reaction of the active support R_N when the body and the rigid plane come into contact. Finally, if there is a reaction R_n i.e. the pressure p_N the spring stiffness was not exploited, contact is achieved and the gap is closed $c(u) = 0$ i.e. $g_n = 0$. In case the spring stiffness has exploited and the system reached its maximum possible deflection, a gap exists between the contact bodies $c(u) > 0$ i.e. $g_n > 0$, then the reaction R_n respectively pressure p_N are zero. The problem with the application of the contact conditions is that the load-displacement functions are not differentiable Fig. 11, it is necessary to apply one of the numerical methods for solving non-differentiable problems as in the mentioned case. One possible method is the Penalty method, which will be discussed in the following section.

3.3. Penalty method

As a solution of contact problems where motion is defined as inequality (32) often applied method in analysis of contact problem with finite elements is application of Penalty method. In case of active restraint, Penalty member is added to equation of potential and the expression is

$$\Pi(u) = \frac{1}{2}ku^2 - mgu + \frac{1}{2}\grave{o}\left[c(u)\right]^2 \quad za \quad \grave{o} > 0 \qquad (42)$$
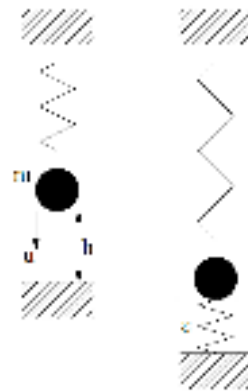

Figure 12. *Point mass supported by a spring and penalty spring due to the penalty term* $\grave{o}$

Penalty parameter ϵ can be interpreted as a spring with definite stiffness that connects contact bodies, in this case mass and rigid surface Fig. 12. The presentation of contact by spring simulation is suitable for reason that energy of Penalty parameter has the same structure as the potential energy of simple spring [1]. The variation of (42) yields for the assumption of contact

$$ku\delta u - mg\delta u - \varepsilon c(u)\delta u = 0 \qquad (43)$$

from which solution for u can be derived as:

$$u = \left(mg + \grave{o}h\right)/\left(k + \grave{o}\right) \qquad (44)$$

According to equation (32), the value of the constraint equation is

$$c(u) = h - u = \frac{kh - mg}{k + \grave{o}} \tag{45}$$

When contact is achieved, according to equation (45) there is $mg \geq kh$ and the "penetration" of point mass into the stiff surface has appeared, which is physically equivalent to compression of spring Fig. 12. Equation of restrain is satisfied only in case when $\grave{o} \to \infty => c(u) \to 0$, penetration depends on penalty parameter.

There are two limiting cases in Penalty method

- $\grave{o} \to \infty => u - h \to 0$ from which follows that the exact solution is got only at big values of Penalty parameter, which means that stiffness of contact spring is very high and only small penetrations occur
- $\grave{o} \to 0$ represents inactive contact or unconstrained solution when Penalty parameter is very small, great penetrations are implied, which leads to problems of convergence

The reaction force for penalty method is computed as:

$$R_N = \grave{o}c(u) = \frac{\grave{o}}{k + \grave{o}}(kh - mg) \tag{46}$$

In the practical computations, large values of the penalty parameter are considered. This leads to the acceptable value of penetration and good calculation results [1]. By restraining parameter $\grave{o} \to \infty$ the same result follows as by applying Lagrange-s multiplier. While using penalty method it is important properly to determine penalty parameter ϵ. If too big penalty parameter is adopted price of calculation significantly rises, and in some cases it can reach oversized stiffness of contact interface and which leads to calculation difficulties. On the other hand, if too small a penalty parameter is adopted the accuracy of results is questionable.

Applying the penalty method on the simple system which is built from two bars with geometrically linear and elastic behavior a nonlinear response curve occurs in the case of contact, Fig. 13. This is due to the change of stiffness within the contact process.
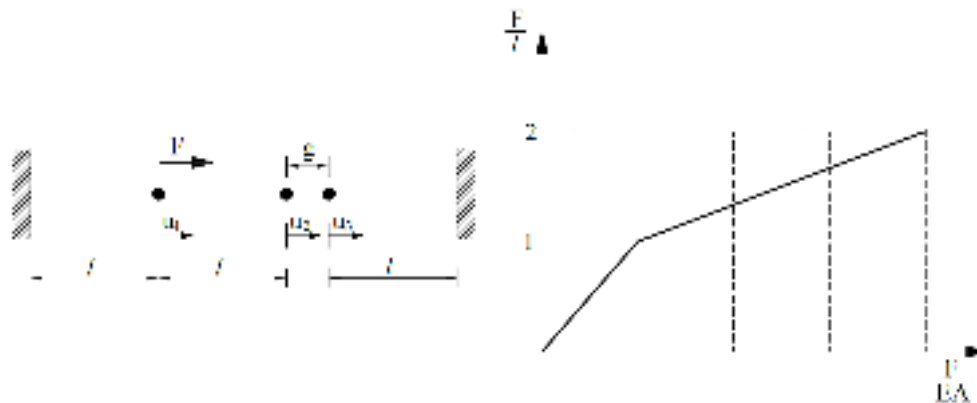

Figure 13. *Finite element discretization of the two separated beams and load-deflection curve*

In general, for a proper definition of penalty parameter, all relevant influences should be considered such as stress level, displacement and nonlinearity in contact interaction. Whether it is a hard or soft contact, is it a mutual contact between two or more bodies or is it a c self-contact, which is the case for large displacements, large rotations and large strains. An example is the deformation of rubber and a polymer under pressure where it is pressed into its own volume or highly deformed steel box profile.

4. CONCLUSION

Although we aimed to present contacts within the framework of small deformations, first we had to define contact problems within the framework of large deformations in order to be able to describe in detail the contact interaction of deformable bodies. Subsequently, certain assumptions were introduced to make the analysis simpler and numerically more acceptable in the first place. Based on the aforementioned we can see how demanding the contact models are and almost always non-linear in nature. From the simple example given in this paper, we saw that even for a system of two simple bars with geometrically linear and elastic behavior a nonlinear response curve is obtained. Finally, we describe the penalty method, which is often used as a basis for other methods. In addition to the penalty method, widely used is the Lagrange multiplier method, which is the basis for the so-called Mortar method of discretization of contact continuous systems within small and large deformations. The Mortar method has been successfully applied and developed for the last 15 to 20 years, we can say that within the mechanics of a deformable body it is still in the development process. In addition to the described methods, for solving the inequality of internal and external virtual works as significant we have: perturbed-Lagrange and augmented-Lagrange method. There are other methods such as the Barrier method and the Nitsche method, but the mentioned methods result in the so-called "ill-conditioning". Due to these drawbacks, the Barrier and Nitsche method are not used very much in computational contact mechanics [1][3].

We can conclude that contact mechanics is a fairly wide range of phenomena to be described and understood, and only for dry friction without the inclusion of adhesion, lubrication, and thermodynamic processes within the contact interaction, which further complicates the analysis and description of contact problem.

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