02| Оригинални научни рад
Research paper
Doi 10.7251/STP1813043M ISSN 2566-4484

PHENOMENA OF INSTABILITY IN NON-CONSERVATIVE DYNAMICAL SYSTEMS

Dragan Milašinović, *ddmil@gf.uns.ac.rs*
University of Novi Sad, Faculty of Civil Engineering, Subotica

Abstract:

If the loading is non-conservative, the loss of stability may not manifest itself as the system going into another equilibrium state, but as exhibiting oscillations of increasing amplitude. To take account of this possibility, we must consider the dynamic behavior of the system, because stability is essentially a dynamic concept. In the paper the author's theory, named the rheological-dynamical analogy (RDA), is used to examine the phenomena of instability in linear internally damped inelastic (LIDI) dynamical systems. Apart from quantitative research, qualitative research is presented to demonstrate the influence of inelasticity and internal friction on dynamic response.

Keywords: non-conservative loading; phenomena of instability; LIDI dynamical systems; RDA.

POJAVA NESTABILNOSTI U NEKONZERVATIVNIM DINAMIČKIM SISTEMIMA

Rezime:

Ako je opterećenje nekonzervativno, gubitak stabilnosti se možda ne manifestuje tako da sistem ide u drugo ravnotežno stanje, nego ispoljava oscilacije povećavajućih amplituda. Da bismo uzeli u obzir ovu mogućnost moramo razmatrati dinamičko ponašanje sistema, jer stabilnost je u suštini dinamički koncept. U ovom radu autorova teorija, pod nazivom reološko-dinamička analogija (RDA), se koristi za ispitivanje pojave nestabilnosti u linearnim interno prigušenim neelastičnim (LIPN) dinamičkim sistemima. Pored kvantitativnog istraživanja, kvalitativno istraživanje je predstavljeno da se pokaže uticaj neelastičnosti i unutrašnjeg trenja na dinamički odgovor.

Ključne reči: nekonzervativno opterećenje; pojava nestabilnosti; LIPN dinamički sistemi; RDA.

1. INTRODUCTION

The characterization of the internal friction and damping capacity of vibrating structures has traditionally been limited in structural dynamics because of the complexity of this problem both in terms of material and structure. The topic of damping has been a problem for quite a long time and it presents severe difficulty in view of physical damping mechanisms. Thus, reliable information on damping is as yet rather scanty [1]. All damping ultimately comes from frictional effects, which may however take place at different scales. But occasionally, the engineer uses damping devices designed to produce beneficial effects. Those devices can often be idealized as lumped objects, modeled as point forces or moments, and said to produce localized damping. One modeling complication is that friction may depend on fabrication or construction details that are not easy to predict; e.g., bolted versus welded connections. Damping models have been criticized by many investigators for various justified reasons and they cannot easily be used without a proper understanding of damping mechanisms [2]. The purpose of this paper is to investigate the stability or instability of LIDI systems due to the initial conditions in the material and various values of the modal damping ratio.

Damping analysis of LIDI systems includes two different classes, one involving the material damping, and the other damping the system under various conditions such as damage and sinusoidal loading. There have been detailed studies into the material damping [3], and also into energy dissipation mechanisms in structural elements [4, 5]. But here difficulty lies in representing these two mechanisms in different parts of the structure in a unified manner. In practice, engineering structures are usually complex, and their dynamic analysis is traditionally performed using the conventional finite element method (FEM). Finite element solutions in dynamics are obtained by employing two different methods [6, 7], the modal method and time marching schemes. In modal analysis, responses of individual modes are superimposed to determine the total response. Traditionally, energy dissipation in a structure is represented as an idealized viscous damping force (non-conservative force), i.e. a force directly proportional to the velocity of the corresponding dynamic system. In this case, the structure mass and stiffness matrices remain constant during the analysis and satisfy the well-known orthogonality conditions. If the damping matrix also satisfies the criterion of orthogonality, the equations of motion for a discretized multiple degree-of-freedom (MDOF) structure can be decoupled into *i* independent equations, one for each normal mode of the structure. Therefore, it is assumed that the modal damping matrix is diagonal, with the modal damping terms $2\xi_i\omega_i$. Ratio ξ_i is defined as the ratio of damping in mode *i* to the critical damping in mode *i*. Consequently, the modal analysis originally developed for undamped systems is used herein to analyze LIDI systems taking into account the viscoelastoplastic (VEP) modal damping ratios [2]. Note that this procedure directly delivers a diagonal modal damping matrix rather than the real damping matrix, which need not be explicitly constructed. To obtain the real damping matrix, mode orthogonality relations must be taken into account.

Nowadays, there is a growing interest in developing a theory which would enable the prediction of non-conservative forces using a unique mathematical formulation, which would include both the material and structural elements. This may be done using the RDA. The RDA inelastic theory has been developed to describe the dynamic response of structures using both the dynamic modulus and modal damping ratios [8, 9]. The dynamic modulus is obtained based on the concept of the complex modulus of VEP materials, whereas the modal damping ratios are obtained by observing critically damped dynamic systems in the steady-state response. It has been proved that the dynamic modulus is equal to the tangent modulus at selected moments in time in some plastic materials [10]. Also, internal damping is a significant factor, considered as a damage variable in low-cycle fatigue. The investigation in the paper [2] shows that internal damping that is unevenly distributed over the elements of a structure causes deterioration of the material named the fatigue damage. The eigenvalues of a structure must first be solved for a dynamic system relieved of external masses, which is required to critically damp it. This is necessary in order to calculate the modal damping ratios for systems composed of consistent or lumped external masses. A system composed of external masses has its own eigenvalues. Also, the RDA is an analytical method whereby resonant frequencies may simply be calculated using the zero modal damping ratios.

The aim of this paper is to demonstrate the validity and applicability of the RDA theory to the problem of the loss of stability of LIDI dynamical systems as non-conservative mechanical systems.

2. VIBRATION AND INELASTIC INSTABILITY

2.1. STEADY-STATE RESPONSE AND THE DAMPING RATIO

The purpose of developing a mathematical model for the rheological behaviour of solids is to permit realistic results to be obtained from mathematical analyses of complicated structures under various conditions, such as sinusoidal, random, and transient loading. Here we consider a damaged long symmetrical rod (e.g., with a square or circular cross-section A_0) of length l_0 and mass density ρ. Let us assume that the load variation is sinusoidal, where Q_A is the amplitude and ω_Q is the frequency. Consider the single degree-of-freedom (SDOF) system of rod shown in Fig. 1.

The continuous model of a rod has only its own mass (m), and it may be modeled as a simple critically damped SDOF system shown in Fig. 1a) using the RDA. Consider the following sinusoidal law of stress,

$$\sigma = \sigma_0 + \sigma_A \sin\left(\omega_\sigma t\right) \tag{1}$$

with σ_0 being a constant and σ_A a variable component of the cycle. $\omega_\sigma = \omega_Q$ is the stress frequency.

The starting point of RDA analysis is the governing differential equation [11],

$$\begin{aligned} \ddot{\varepsilon} m + \dot{\varepsilon} c + \varepsilon k = \sigma_A \left(\frac{k}{E_H} + \frac{E_K + H'}{\gamma} - \omega_\sigma^2 \frac{m}{E_H} \right) \sin\left(\omega_\sigma t\right) + \\ + \sigma_A \left(\frac{c}{E_H} + \frac{\lambda_K + \lambda_N}{\gamma} \right) \omega_\sigma \cos\left(\omega_\sigma t\right) + \sigma_0 \left(\frac{k}{E_H} + \frac{E_K + H'}{\gamma} \right) - \sigma_Y \frac{E_K}{\gamma} \end{aligned} \tag{2}$$

where E_H is the elastic modulus, σ_Y the uniaxial yield stress, and $Y = \sigma_Y + H'\varepsilon_{vp}\left(t\right)$ the VEP yield condition. The four properties at fixed step times are the extensional viscoelastic viscosity λ_K, the extensional viscoplastic viscosity λ_N, the viscoelastic modulus E_K, and the viscoplastic modulus H'.

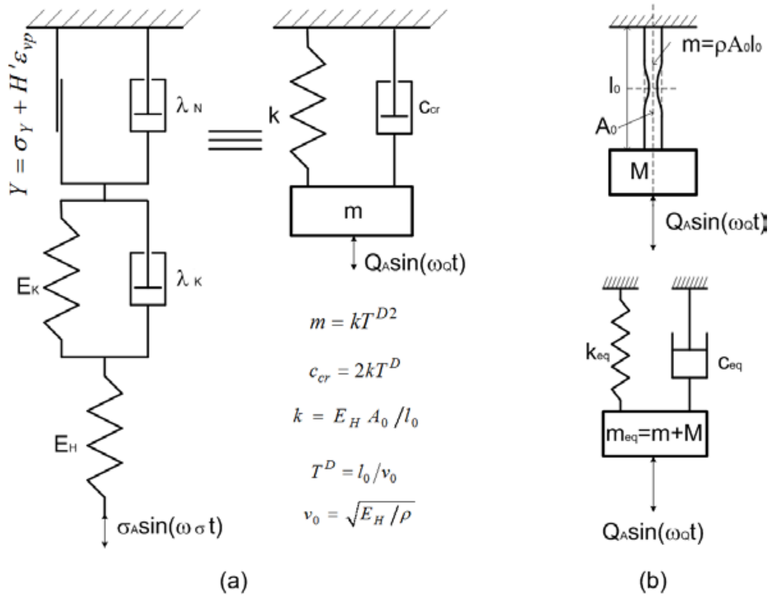

Figure 1. Models of rods: (a) a distributed critically damped RDA model; (b) an SDOF system

The particular solution (steady-state response) of the second-order governing equation in the case of critical damping ($E_K/\lambda_K = H'/\lambda_N$, $\lambda_K = E_K T_K$, $\lambda_N = H'T^*$, $T_K = T^* = T^D$) can be written as [12],

$$\varepsilon_p(t) = \frac{\sigma_A}{E_H}\sqrt{\frac{(1+\varphi)^2+\delta^2}{1+\delta^2}}\sin\left(\omega_\sigma t - \arctan\frac{\delta\varphi}{1+\delta^2+\varphi}\right) \tag{3}$$

where φ is the creep coefficient, $\delta = \omega_Q/\omega = T^D\omega_Q$ the relative frequency, and T^D the delay time.

First, we define a conservative mechanical system as one whose generalized forces are completely derivable from the potential energy function $Q_\alpha = -\partial V/\partial q^\alpha$ (principle of virtual work). Further, for non-conservative systems, non-conservative forces must be added. A discrete linear SDOF system is characterized by one resonant frequency, and it may comprise a single element of mass and one or more elements of stiffness, or a single element of stiffness and one or more elements of mass. Consequently, we can form a non-conservative SDOF system, as shown in Fig. 1b), which consists of its own mass and an attached external mass *M*, with the following equation of motion,

$$m_{eq}\ddot{u} + c_{eq}\dot{u} + k_{eq}u = Q_A\sin(\omega_Q t), \tag{4}$$

where

$$m_{eq} = m + M = m(1+\eta),\ c_{eq} = \xi c_{cr} = 2\xi\sqrt{k_{eq}m_{eq}},\ k_{eq} = E_R A_0/l_0\,. \tag{5}$$

m_{eq} is the equivalent mass, where $\eta = M/m$ is the mass ratio, whereas the equivalent damping is denoted by c_{eq}, with ξ as the damping ratio. k_{eq} is the equivalent axial stiffness, through which the internal damping is included taking into account the dynamic modulus [10],

$$E_R = E_H \frac{1+\delta^2+\varphi}{(1+\varphi)^2+\delta^2}, \quad \underset{\lim\delta\to 0}{E_R} = E_H \frac{1}{1+\varphi} \tag{6}$$

The natural ω^* and relative δ^* frequencies, respectively, are as follows,

$$\omega^* = \sqrt{\frac{k_{eq}}{m_{eq}}} = \omega \frac{1}{\sqrt{(1+\varphi)(1+\eta)}}, \tag{7}$$

$$\delta^* = \frac{\omega_Q}{\omega^*} = \frac{\omega_Q}{\omega}\sqrt{(1+\varphi)(1+\eta)} = \delta\sqrt{(1+\varphi)(1+\eta)}. \tag{8}$$

The well-know particular solution (steady-state response) is given by

$$u_p(t) = A\sin\left(\omega_Q t - \arctan\left(\frac{2\xi\delta^*}{1-\delta^{*2}}\right)\right). \tag{9}$$

Dynamic measurements supply information not only on the dynamic modulus E_R, but also on the phase or loss angle α. The loss angle is a measure of the amount of energy dissipated in the material during one cycle. The variation of the loss angle with frequency δ is shown in Fig. 2.

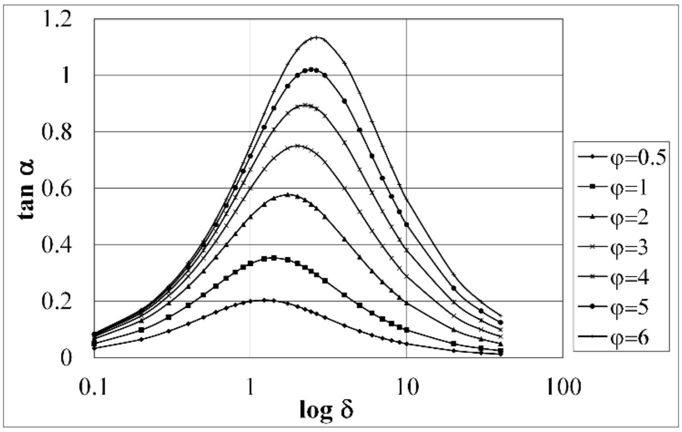

Figure 2. Variation of the loss angle with relative frequency δ

According to the principle of analogy [8], the phase angle at a point of a continuum must be described by both the critically damped RDA model and the corresponding SDOF system in the steady-state response. As a result, we can form an equality in order to obtain the VEP damping ratio based on the phase angle.

$$\frac{\delta\varphi}{1+\delta^2+\varphi} = \frac{2\xi\delta^*}{1-\delta^{*2}} \Rightarrow \xi = \frac{\delta\varphi\left(1-\delta^{*2}\right)}{2\delta^*\left(1+\delta^2+\varphi\right)}. \tag{10}$$

Fig. 3 (left) presents the modal damping ratio as a function of relative frequency δ for the creep coefficient $\varphi = 2$ and three mass ratios η. The dependence of ξ on δ^* shows that ξ has negative values for $\delta^* \rangle 1$. As shown in Fig. 3 (right), the smaller the mass ratio, the greater the modal damping ratio. Similarly, the greater frequency δ^*, the smaller the modal damping ratio.

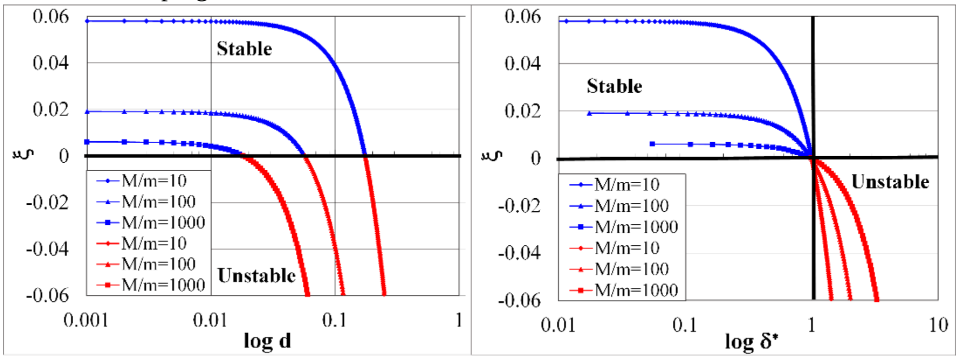

Figure 3. Variation of the VEP damping ratio with frequencies δ (left) and δ (right)*

2.2. COMPLEMENTARY SOLUTIONS

The usual form of the complementary solution is

$$u_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}, \tag{11}$$

where C_1 and C_2 are arbitrary constants determined by the initial conditions imposed on the system, and r_1 and r_2 are the roots of the auxiliary equation $m_{eq}\ddot{u} + c_{eq}\dot{u} + k_{eq}u = 0$, $\omega^* = \sqrt{k_{eq}/m_{eq}}$ and $\xi = c_{eq}/2m_{eq}\omega^*$. Consequently,

$$r_1 = \omega^*\left(-\xi + \sqrt{\xi^2 - 1}\right), \quad r_2 = \omega^*\left(-\xi - \sqrt{\xi^2 - 1}\right). \tag{12}$$

These values of r may be real and distinct, real and equal, or complex conjugates, depending on the magnitudes of ξ.

2.2.1. Non-oscillatory stable motion, $\xi > 1$

If ξ is greater than unity, the values of r are real and negative. Therefore, no oscillatory motion is possible according to the complementary solution of the equation of motion, regardless of the initial conditions u^0 and $\dot{u}^0$ imposed on the system. This is a case of overdamping of the system, where

$$u_h(t) = e^{-\xi\omega^* t}\left[u^0 \cosh\hat{\omega}t + \frac{1}{\hat{\omega}}\left(\dot{u}^0 + u^0\xi\omega^*\right)\sinh\hat{\omega}t\right], \tag{13}$$

and

$$\hat{\omega} = \omega^* \sqrt{\xi^2 - 1} \ . \tag{14}$$

2.2.2. Non-oscillatory critically damped stable motion, $\xi = 1$

If ξ is equal to unity, the values of *r* are equal to $-\omega^*$. Again, the motion is not oscillatory, and its amplitude will eventually diminish to zero. Here, the system is critically damped, where

$$u_h\left(t\right) = \left[u^0 + \left(\dot{u}^0 + \omega^* u^0\right)t\right]e^{-\omega^* t} \ . \tag{15}$$

2.2.3. Stable motion oscillates about the equilibrium configuration and decays toward it, $1 > \xi > 0$

If ξ is less than unity, the values of *r* are complex conjugates. They are then

$$r_1 = \omega^*\left(-\xi + i\sqrt{1-\xi^2}\right), \quad r_1 = \omega^*\left(-\xi - i\sqrt{1-\xi^2}\right). \tag{16}$$

The complementary solution takes the following form,

$$u_h\left(t\right) = Be^{-\xi\omega^* t}\cos\left(\omega^d t - \arctan\left(\frac{\dot{u}^0 + u^0 \xi \omega^*}{u^0 \omega^d}\right)\right). \tag{17}$$

where

$$B = \sqrt{\left(u^0\right)^2 + \left(\frac{\dot{u}^0 + u^0 \xi \omega^*}{\omega^d}\right)^2} \ , \tag{18}$$

and

$$\omega^d = \omega^* \sqrt{1-\xi^2} \ . \tag{19}$$

This is a harmonic motion of the damped natural frequency ω^d, the amplitude $Be^{-\xi\omega^* t}$ of which decreases exponentially with time.

2.2.4. Harmonic oscillatory undamped stable motion, $\xi = 0$

The equation of motion of a dynamical system that is relieved of external masses ($\eta \to 0$) and made from an idealized purely elastic material ($\varphi \to 0$) becomes $m\ddot{u} + ku = 0$. Two initial conditions must appear in the general solution,

$$u_h\left(t\right) = u^0 \cos \omega t + \frac{\dot{u}^0}{\omega} \sin \omega t \ . \tag{20}$$

Physically, this equation represents an undamped free vibration. The angular natural frequency is

$$\omega = \sqrt{k/m} \ . \tag{21}$$

2.2.5. Unstable motion oscillates about and grows toward another equilibrium configuration, $-1 < \xi < 0$

If ξ is between -1 and 0, the complementary solution is given by

$$u_h(t) = e^{-\xi\omega^* t}\left(u^0 \cos\omega^d t + \frac{1}{\omega^d}\left(\dot{u}^0 + u^0\xi\omega^*\right)\sin\omega^d t\right). \qquad (22)$$

This is a harmonic unstable motion, which oscillates about and grows toward another equilibrium configuration.

2.2.6. Unbounded non-oscillatory unstable motion, $\xi < -1$

If ξ is less than -1, the complementary solution is in terms of hyperbolic functions,

$$u_h(t) = e^{-\xi\omega^* t}\left[u^0 \cosh\hat{\omega}t + \frac{1}{\hat{\omega}}\left(\dot{u}^0 + u^0\xi\omega^*\right)\sinh\hat{\omega}t\right]. \qquad (23)$$

This is non-oscillatory motion, and it is unstable and unbounded.

2.3. DYNAMIC CRITERIA FOR THE STABILITY AND INSTABILITY OF MOTION BY DAMPING RATIO

For the analysis of the stability of a mechanical system, the method of small vibrations is commonly used. The method entails the derivation of equations of motion for small displacements from the equilibrium state. Small displacements make it possible to take into account only those terms which are homogeneous and linear in the displacements or their derivatives. The homogeneous and linear equations derived in this manner have solutions whose dependence on time is characterized by their common factor e^{pt}.

The Liapunov definition of stability can thus be informally written as follows: *The equilibrium state under consideration is stable if and only if, for all solutions of this form, the real part of p is non-positive. Otherwise, it is unstable.*

All real dynamical systems in physics and engineering are characterized by stable steady state vibration, which involves the dissipation of some energy as heat, even if negative damping occurs, because of the member $(2\xi\delta)^2$ in the steady-state solution of vibration. In case of the negative damping ratio, the amplitude of complementary vibration, which is multiplied with member e^{pt}, will not subside due to this member. A mechanical system with a positive damping ratio can be called a dynamically stable system, whereas one with a negative ratio can be called dynamically unstable. Consequently, the dynamic stability and instability of mechanical LIDI systems can be defined using the damping ratio, Table 1.

Table 1. Dynamic criteria for the stability and instability of motion according to the daming ratio.

$\xi > 1$	Non-oscillatory stable motion
$\xi = 1$	Non-oscillatory critically damped stable motion
$1 > \xi > 0$	Stable motion oscillates about the equilibrium configuration and decays toward it
$\xi = 0$	Harmonic oscillatory undamped stable motion
$-1 < \xi < 0$	Unstable motion oscillates about and grows toward another equilibrium configuration
$\xi < -1$	Unbounded non-oscillatory unstable motion

Fig. 4 shows various types of motion as determined by the damping ratio.

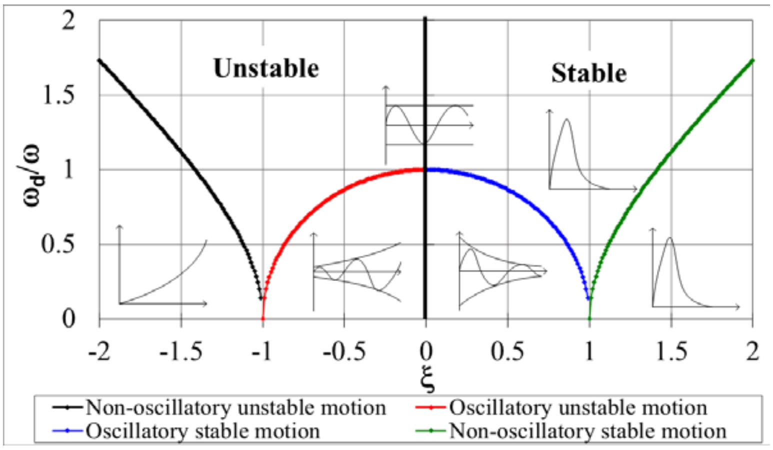

Figure 4. Various types of motion determined according to the damping ratio

3. RHEOLOGICAL-DYNAMICAL ASPECTS OF DAMAGE

Once the VEP damping ratio is determined, it can be formulated as a function of the greatest value of the loss angle,

$$\xi = \max \tan \alpha \frac{1-\delta^{*2}}{\sqrt{1+\eta}\left(1+\delta^2+\varphi\right)}. \tag{24}$$

Let us now consider the quasi-static loading ($\omega_Q \to 0$) of an LIDI system that is relieved of external masses ($\eta \to 0$),

$$\xi = \max \tan \alpha \frac{1}{1+\varphi}. \tag{25}$$

In this case the damping ratio depends of the creep coefficient only. As the creep coefficient φ is always higher than zero in accordance with the second law of thermodynamics, the damping ratio has a positive value. Therefore, a dynamical system is stable if quasi-static loading is applied.

The RDA approach used to consider both the initial (undamaged) and damaged state of the cylindrical rod for the analysis of the influence of Poisson’s ratio on the creep coefficient has already been described in [12]. Fig. 5 presents a relation whose results are in excellent agreement with the experimentally obtained values.

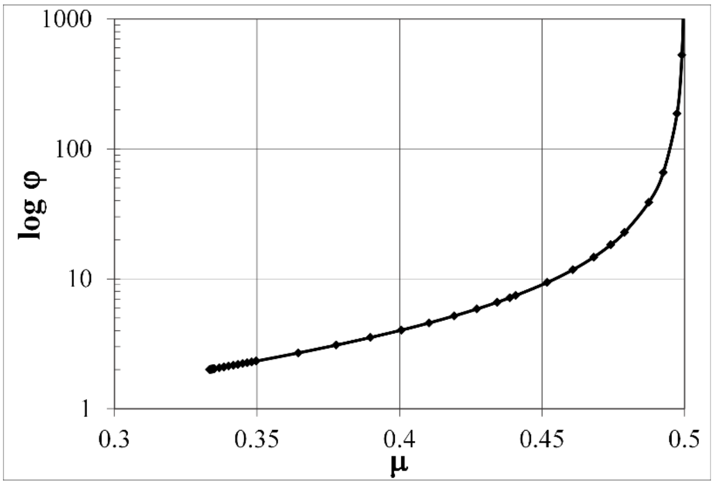

Figure 5. Variation of the creep coefficient with Poisson's ratio

Using the RDA inelastic theory, based on Bernoulli's energy theorem and assuming that $\varepsilon_E = \sigma_E/E_H = 0.001$, the creep coefficient is expressed by the formula

$$\varphi(\mu) = \left[\left(\frac{1}{1-0.001\mu}\right)^4 - 1\right]\frac{1}{2\cdot 0.001} \Bigg/ \left\{1 - \left[\left(\frac{1}{1-0.001\mu}\right)^4 - 1\right]\frac{1}{2\cdot 0.001}\right\}. \quad (26)$$

Eq. (26) can be simplified by neglecting the products of second-order exponents,

$$\left[\left(\frac{1}{1-0.001\mu}\right)^4 - 1\right]\frac{1}{2\cdot 0.001} = \frac{2\mu}{1-0.004\mu} \approx 2\mu. \quad (27)$$

Hence, the mathematical expression for obtaining the value of the creep coefficient from Poisson's ratio is

$$\varphi = \frac{2\mu}{1-2\mu}. \quad (28)$$

Poisson's ratio as defined in an idealized purely elastic material is an elastic constant. However, such a material is hypothetical because it does not exist in the strict sense of the word. In any mechanical deformation the deformation energy is not only stored elastically, but part of it is invariably dissipated by viscous forces, in accordance with the second law of thermodynamics. This dissipation is responsible for the time dependence of the mechanical properties of any real material. Consequently, the VEP Poisson ratio in the time domain chosen can be defined as suggested in [13],

$$\mu(t) = \mu_e - \left(\mu_e - \mu_g\right)e^{-\frac{t}{T^D}}, \quad (29)$$

where $\mu_e = 0.5$ and $\mu_g = 0.333$ are the equilibrium and instantaneous Poisson ratios, respectively.

Experiments with ductile mild steel show that the measured value of Poisson's ratio corresponds to the instantaneous value of 0.333. An axial fatigue experiment was performed on an isolated reinforced steel rod, l_0=50 cm, Φ=1.9 cm [12]. T^D = 0.0000967s is the delay time for the steel rod (prototype). It has been proved that the limit of elasticity of ductile mild steel is in good accordance with the slenderness ratio of $l_0/i = 105.26$, where $i = \sqrt{I/A_0} = \Phi/4 = 0.475cm$. The critical stress at the limit of elasticity of the two-hinged rod is defined by Euler's formula $\sigma_E = \pi^2 E_H / (l_0/i)^2 = 187MPa$, where E_H = 210000 MPa. Fig. 6 presents Poisson's ratio as a function of time for the steel rod (prototype).

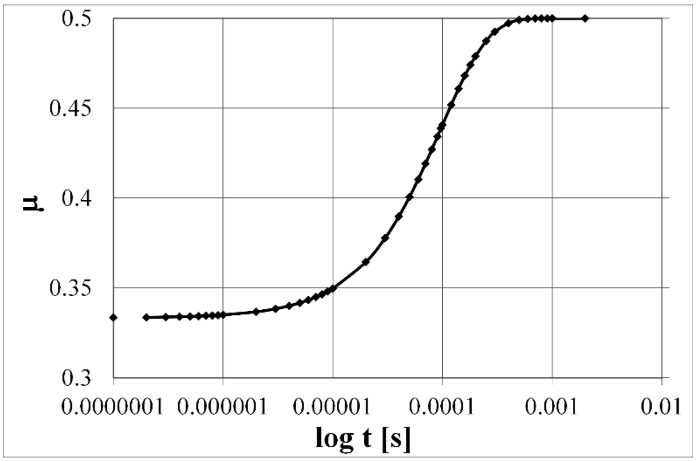

Figure 6. Poisson's ratio versus time for the steel rod (prototype)

At the limit of elasticity, the following equality has already been derived in [11],

$$\frac{\pi^2}{(l_0/i)^2} = \frac{\gamma\varphi}{(l_0/i)(i^3/I)}. \tag{30}$$

This is the intersection of Euler's and RDA buckling curves, which is important for determining the creep coefficient. Hence,

$$\varphi = \pi^2 \left(i^3/I\right) \frac{1}{\gamma(l_0/i)} = \pi^2 \cdot 0.1675315 \frac{1}{7.86 \cdot 10^{-3} \cdot 105.26} = 2.$$

The following symbols are used in the above expression, $i^3/I = 1/(\Phi \cdot \pi) = 0.1675315$ and γ, which is the specific gravity of steel. The value of the creep coefficient corresponds exactly to the value that can be obtained from Eq. (28) using $\mu_g = 0.333$,

$$\varphi_g = \frac{2\mu_g}{1-2\mu_g} = 2.$$

Knowing the value of $\mu(t)$, we can calculate the change in the volume of the rod under tension or compression. The relative change in the volume resulting from uniaxial extension is as follows [13],

$$\frac{\Delta V(t)}{V_0} = (1-2\mu)\varepsilon - \mu(2-\mu)\varepsilon^2 + \mu\varepsilon^3 \,. \tag{31}$$

In the above equation the viscoelastic (VE) strain in the case of critical damping is given as follows [11],

$$\varepsilon = \frac{\sigma_E}{E_H}\left[1+\varphi\left(1-e^{-\frac{t}{T^D}}\right)\right]. \tag{32}$$

Thus, $\Delta V(t)=0$ yields the traditional incompressibility relation for Poisson's ratio, $\mu_e = 0.5$, only if all higher strain terms are neglected. However, since for the majority of materials $\mu\langle 0.5$, tension is accompanied by an increase in the volume, and compression by a decrease in the volume. The change in the volume of the steel rod (prototype) with Poisson's ratio is shown in Fig. 7, where

$$\gamma(t) = 1 + \frac{\Delta V(t)}{V_0}. \tag{33}$$

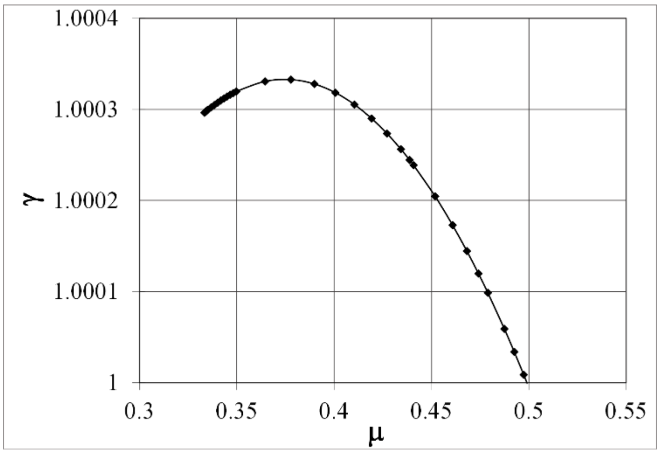

Figure 7. Relative change of volume versus Poisson's ratio for the steel rod (prototype)

Fig. 8 presents the variation of the greatest value of the loss angle with Poisson's ratio according to the expression

$$\max \tan\alpha = \mu\sqrt{\frac{1}{1-2\mu}} \,. \tag{34}$$

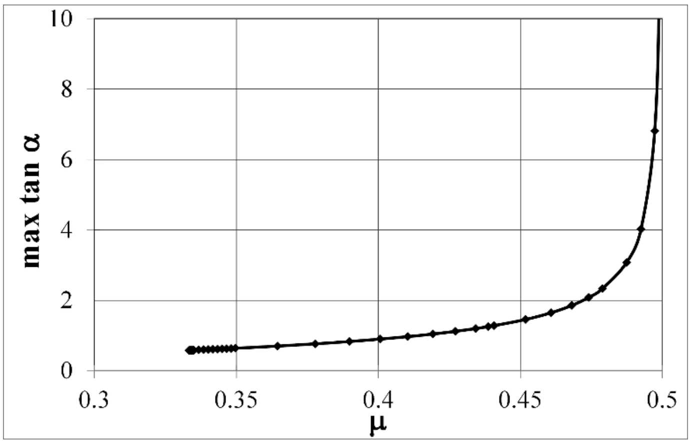

Figure 8. Greatest value of the loss angle versus Poisson's ratio

The relationship of the damping ratio versus Poisson's ratio is given by

$$\xi = \mu\sqrt{1-2\mu}\ . \tag{35}$$

The variation of the VEP damping ratio with Poisson's ratio is shown in Fig. 9.

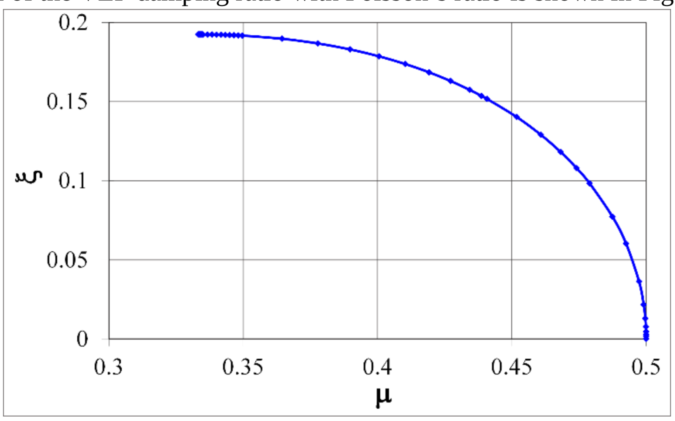

Figure 9. Damping ratio versus Poisson's ratio

The above analysis shows that any real material at the limit of elasticity is a VE material whose $\varphi \neq 0$ and which must thus be treated as damaged. Since the development of micro cracks induces a reduction in the stiffness of the material, the damage state can also be characterized by a variation in the elastic modulus [14]. If we suppose that the variation from an undamaged to a damaged state is equal to the dynamic modulus E_R, the damage variable can be defined as follows [15],

$$D = \frac{(1+\varphi)\varphi}{(1+\varphi)^2 + \delta^2}, \quad \underset{lim\delta\to 0}{D} = \frac{\varphi}{1+\varphi}. \tag{36}$$

Hence, the damage is described by a scalar variable *D*, which ranges between 0.666 and 1 in the case of the steel rod (prototype), Fig. 10.

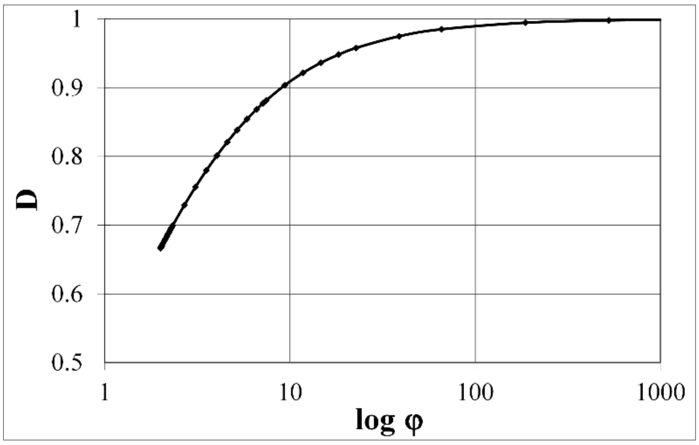

Figure 10. Damage variable versus the creep coefficient

Finally, Fig. 11 presents the variation of the VEP damping ratio with the damage variable according to the expression

$$\xi = \frac{D}{2}\sqrt{1-D}\,. \tag{37}$$

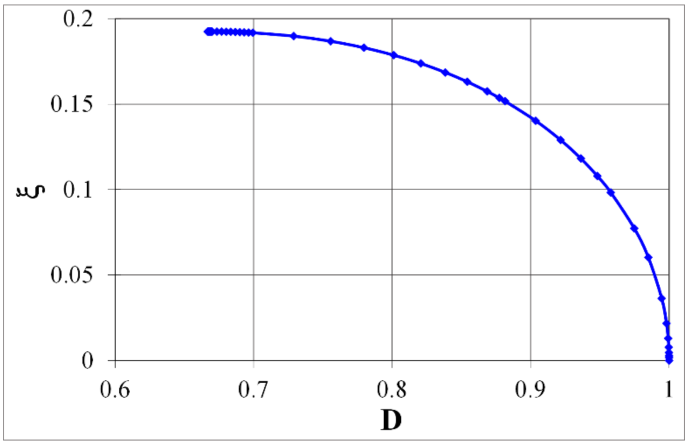

Figure 11. Damping ratio versus the damage variable

It is clearly seen from the above expressions that the values of the VE Poisson ratio play an important role in dictating the response in a given medium. Poisson's ratio is an elastic constant defined as the ratio of the lateral contraction to the elongation in the infinitesimal uniaxial extension of a homogeneous isotropic body. In a VEP material Poisson's ratio is a function of time [13]. The damping ratio decreases with time in a stable LIDI system if quasi-static loading is applied. Energy is usually dissipated as heat due to viscous forces. Therefore, the motion oscillates about the equilibrium configuration and decays toward it, whereas the total energy in the system decreases with time. Also, the RDA improves the prediction of instantaneous values of mechanical parameters at the limit of elasticity, which are defined by Poisson's ratio only, i.e. $\mu_g = 0.333$, $\varphi = 2$, $\xi = 0.19245$, $D = 0.666$ and $E_R = E_H/3$.

4. PHENOMENA OF INELASTIC INSTABILITY BASED ON THE FREQUENCY OF EXCITATION

Resonance occurs when the damping ratio is zero, i.e. $\delta^* = \omega_Q / \omega^* = 1$, and it can be expressed as follows,

$$\omega_Q = \omega^* = \omega \frac{1}{\sqrt{1+\varphi}}. \tag{38}$$

When this happens, the amplitude of vibration increases without bound. In this case, motions are neutrally stable and present a transition between stable and unstable motions.

Fig. 12 presents the VEP damping ratio as a function of relative frequency, according to the expression

$$\xi = \frac{\varphi}{\sqrt{1+\varphi}}\left(1-\delta^{*2}\right) \Big/ 2\left(1+\frac{\delta^{*2}}{1+\varphi}+\varphi\right). \tag{39}$$

Three Poisson's ratios are used in this analysis. The dependence of ξ on δ^* shows that the VEP damping ratio has negative values for $\delta^* \rangle 1$. It means that the corresponding motion is unstable (marked with red).

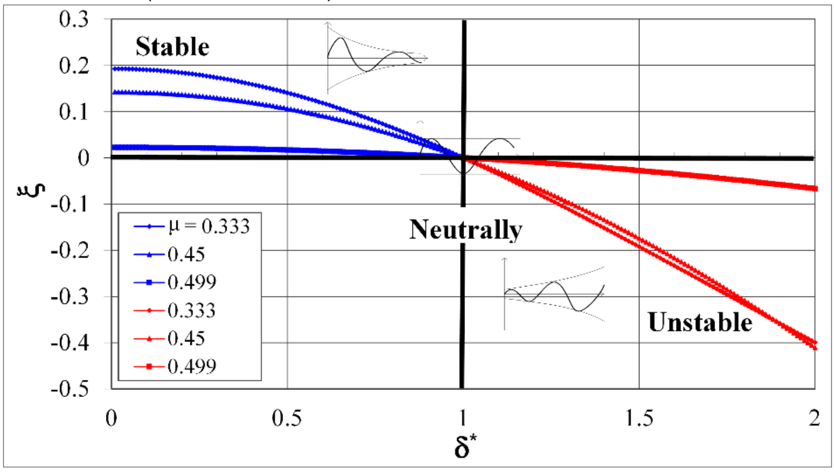

Figure 12. Damping ratio versus frequency δ^*: stable motion (blue) and unstable motion (red)

For the unstable LIDI system, energy must have been kept and added to the system because of the continuous increase in the amplitude of vibration. In this case, work is performed on a linear system by a viscous force due to the frequency of excitation, which must be higher than the resonant frequency. Therefore, the VEP damping ratio for the unstable system must be negative. Unstable motion oscillates about the equilibrium configuration and grows toward another configuration.

Fig. 13 left shows the VEP damping ratio as a function of Poisson's ratio using three relative frequencies. Two motions are described with two lines, stable motion ($\delta^* = 0$, marked with blue), and unstable motion ($\delta^* = 2$, marked with red). The neutrally stable or resonant motion ($\delta^* = 1$) is marked with black. The highest value of the damping ratio ($\xi = 0.19245$) corresponds to the instantaneous Poisson ratio ($\mu_g = 0.333$) in stable motion. Fig. 13 right presents the VEP damping ratio as a function of the creep coefficient. The highest value of the damping ratio corresponds to the instantaneous creep coefficient ($\varphi = 2$).

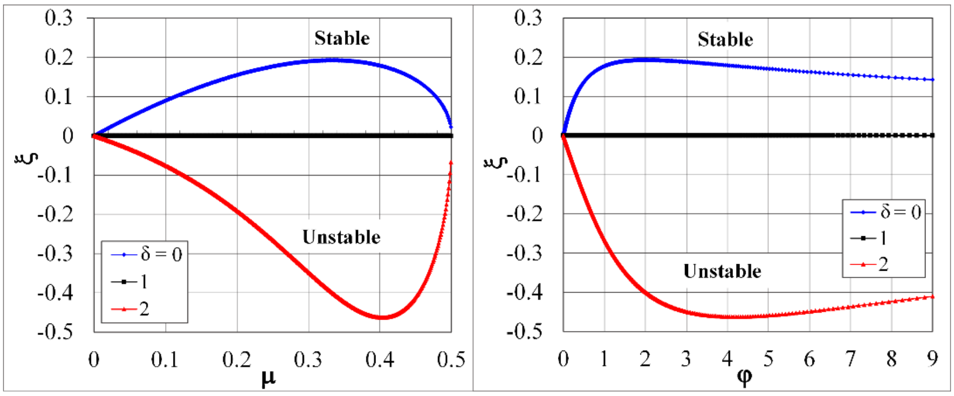

Figure 13. Damping ratio versus Poisson's ratio (left) and damping ratio versus the creep coefficient (right): stable motion (blue), unstable motion (red), and neutrally stable motion (black)

Consequently, the dynamic stability and instability of an LIDI mechanical system can be defined according to the value of the frequency of excitation, Table 2.

Table 2. Dynamic criteria for the stability and instability of motion based on the frequency of excitation.

$\omega_Q \langle \omega^*$	$\delta^* \langle 1$	$\xi \rangle 0$	Stable motion oscillates about the equilibrium configuration and decays toward it.
$\omega_Q = \omega^*$	$\delta^* = 1$	$\xi = 0$	Neutrally stable or resonant motion.
$\omega_Q \rangle \omega^*$	$\delta^* \rangle 1$	$-1 \langle \xi \langle 0$	Unstable motion oscillates about and grows toward another equilibrium configuration.
$\omega_Q \rangle \omega^*$	$\delta^* \rangle 1$	$\xi \langle -1$	Unbounded non-oscillatory unstable motion.

It should be noted that in the case of quasi-static loading, i.e. when the frequency of excitation is zero, the VEP damping ratio always has a positive value, and the LIDI system is stable. However, increasing the frequency of excitation even insignificantly may make the system unstable if damage growth is nonetheless sufficient, or if external mass is added. Otherwise, in the case of dynamic loading and initial conditions in the material, when there is no growth of damage and no external mass is added, the main cause of instability is a rise of the frequency of excitation.

5. CONCLUSIONS

This paper presents the use of the RDA for a systematic study of the vibration and stability of LIDI dynamical systems. The current literature appears to ignore the true nature of damping mechanisms. Summarized below are the paper conclusions, with recommendations for work in areas which have so far been neglected. The following points are emphasized:

- In any mechanical deformation the deformation energy is not only stored elastically, but part of it is invariably dissipated by viscous forces, in accordance with the second law of thermodynamics. This dissipation is responsible for the time dependence of the mechanical properties of any real material.
- In VEP materials Poisson's ratio is a function of time that depends on the time regime chosen to elicit it. The function that has been suggested in [13] is used in this paper for the analysis of the steel rod (prototype). It is proved that the RDA improves the prediction of instantaneous values of mechanical parameters at the limit of elasticity.
- The RDA approach has already been used for the analysis of the influence of Poisson's ratio on the creep coefficient [12]. Hence, if the creep coefficient is to be used in conjunction with Poisson's ratio, it must be determined using Eq. (28).
- Because of the analogy, a critically damped RDA model has the same phase angle as the equivalent SDOF system in the steady-state response [8]. Therefore, if the VEP damping ratio is to be used in conjunction with the creep coefficient, it must be determined using Eq. (10).
- The dynamic stability or instability of an LIDI system can be defined as follows:

  $\xi \rangle 0$; Stable motion oscillates about the equilibrium configuration and decays toward it.

  $\xi = 0$; Neutrally stable or resonant motion.

  $-1 \langle \xi \langle 0$; Unstable motion oscillates about and grows toward another equilibrium configuration.

  $\xi \langle -1$; Unbounded non-oscillatory unstable motion
- The RDA is very efficient when applied to inelastic MDOF systems, because it reduces a material non-linear problem to a linear dynamic problem, which allows the use of modal analysis. On the other hand, the classical solutions are not in accordance with the actual damping mechanisms of LIDI systems. Note that this procedure assumes a constant damping factor typically lower than 10%.
- In the case of quasi-static loading, i.e. when the frequency of excitation is zero, the VEP damping ratio is always positive and the LIDI system stable.
- In the case of dynamic loading and initial conditions in the material, the main cause of instability is a rise of the frequency of excitation. However, for frequencies that are smaller than the smallest resonant frequency of the LIDI system, the motions of all DOF will be stable.

Acknowledgements
The work presented in this paper is a part of the investigation conducted within the research projects ON 174027 "Computational Mechanics in Structural Engineering" and TR 36017 "Utilization of by-products and recycled waste materials in concrete composites for sustainable construction development in Serbia: investigation and environmental assessment of possible applications", supported by the Ministry of Science and Technology of the Republic of Serbia. This support is gratefully acknowledged.

LITERATURE (Use style LITERATURE)

[1] Adhikari S, and Woodhouse J. “Identification of damping: Part 1, Viscous damping” Journal of Sound and Vibration, vol. 243, no. 1, 2001, pp. 43–61.
[2] Milašinović DD, and Landović A. “Rheological-Dynamical analogy for analysis of vibrations and low cycle fatigue in internally damped inelastic frame structures.” Computers and Structures , vol. 196, 2018, pp. 76–93.
[3] Bert CW. “Material damping: An introductory review of mathematic measures and experimental technique.” Journal of Sound and Vibration, vol. 29, no. 2, 1973, pp. 129–153.
[4] Veletsos AA, and Ventura CE. “Modal analysis of non-classically damped linear systems.” Earthquake Engineering and Structural Dynamics, vol. 14, no. 2, 1986, pp. 217–243.
[5] Abdelraheem Farghaly a. “Parametric study on equivalent damping ratio of different composite structural building systems.” International Journal of Steel Structures, vol. 15, no. 1, 2015, pp. 7–16.
[6] Bathe KJ. Finite element procedures. NY (USA): Prentice Hall, 1996.
[7] Chopra AK. Dynamics of structures: theory and applications to earthquake engineering. NY (USA): Prentice Hall, 1996.
[8] Milašinović DD. “Rheological-dynamical analogy: design of viscoelastic and viscoplastic bar dampers.” Mech Time-Depend Mater, vol. 14, no. 4, 2010, pp. 389–409.
[9] Milašinović DD. “Rheological-dynamical analogy: prediction of damping parameters of hysteresis damper.” Int J Solids Struct , vol. 44, no. 22-23, 2007, pp. 7143–7166.
[10] Milašinović DD. “Rheological-dynamical analogy: visco-Elasto-Plastic behavior of metallic bars.” Int J Solids Struct , vol. 41, no. 16-17, 2004, pp. 4599–4634.
[11] Milašinović DD. “Rheological-dynamical analogy: prediction of buckling curves of columns.” Int J Solids Struct , vol. 37, no. 29, 2000, pp. 3965–4004.
[12] Milašinović DD. “Rheological-dynamical analogy: modeling of fatigue behavior.” Int J Solids Struct , vol. 40, no. 1, 2003, pp. 181–217.
[13] Tschoegl NW, et al. “Poisson’s ratio in linear viscoelaticity - a critical review.” Mech Time-Depend Mater, vol. 6, 2002, pp. 3–51.
[14] Lemaitre J. “How to use damage mechanics.” Nucl Eng Des , vol. 80, 1984, pp. 233–245.
[15] Milašinović DD. “Rheological-dynamical continuum damage model for concrete under uniaxial compression and its experimental verification.” Theoret Appl Mech, vol. 42, no. 2, 2015, pp. 73–110.